§3.6 Permutation Groups

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MATH 546/701I

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- For small n, check $\mathbf{Z}_n^\times$ cyclic or not without using *primitive root thm*.

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- *Notation*: $\mathrm{Sym}(S) = \{\sigma \mid \sigma : S \to S\}$ or write S_n if $S = \{1, 2, \ldots, n\}$.
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- Let $\sigma \in \mathrm{Sym}(S)$. Then $\sigma = (a_1 a_2 \cdots a_k)$ is a cycle of length k.
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In fact, this natural way will be important in the proof of Cayley's theorem.

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So, $G \cong G_\lambda$, where G_λ is a subgroup of $\mathrm{Sym}(G)$, i.e., a permutation group.

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a change in position where the distance between points is preserved and figures remain congruent (having the same size and shape). It may be

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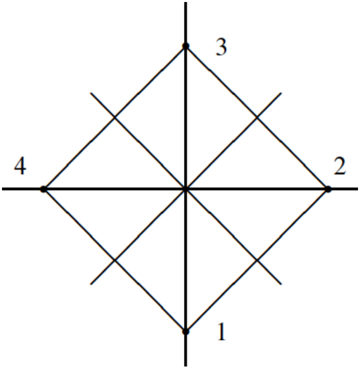
Example cont.: Rigid motions of a square



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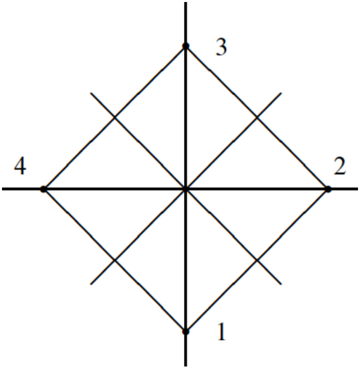

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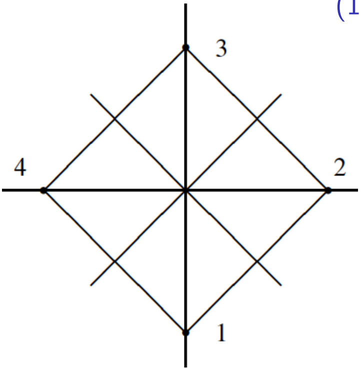

(1234) counterclockwise rotation through 90°

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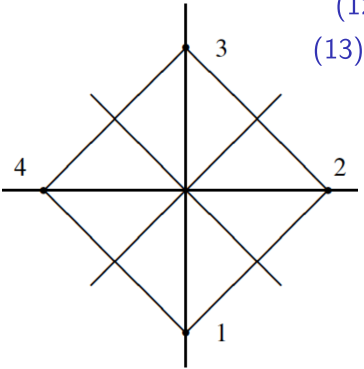

(1234)	counterclockwise rotation through 90°
(13)(24)	counterclockwise rotation through 180°

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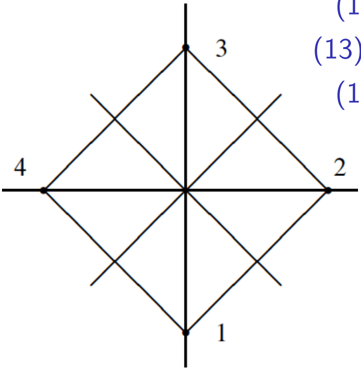

(1234)	counterclockwise rotation through 90°
(13)(24)	counterclockwise rotation through 180°
(1432)	counterclockwise rotation through 270°

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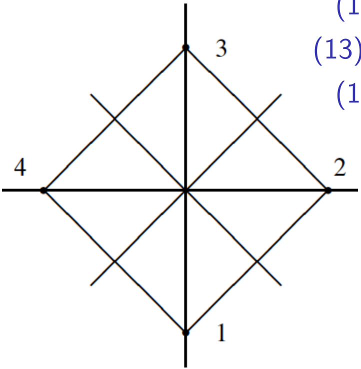

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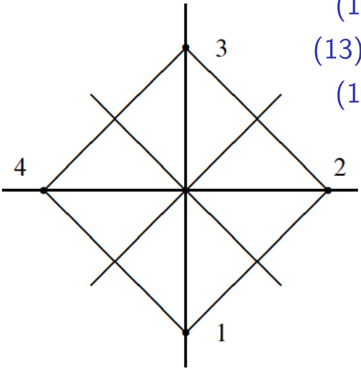

(1234) counterclockwise rotation through 90°

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(24) flip about vertical axis

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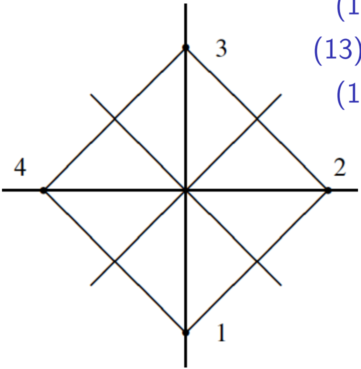

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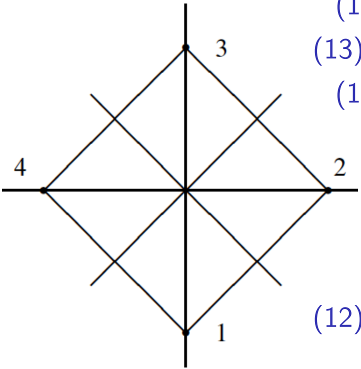

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(13) flip about horizontal axis
(12)(34) flip about diagonal

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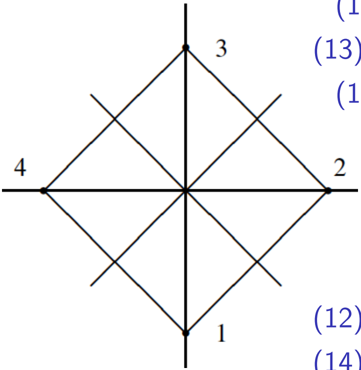

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(12)(34)	flip about diagonal
(14)(23)	flip about diagonal

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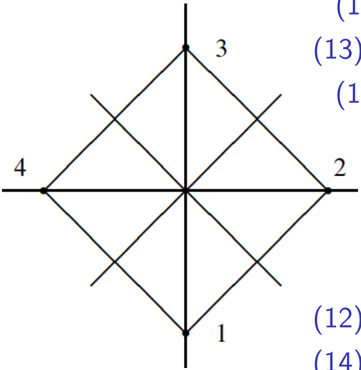

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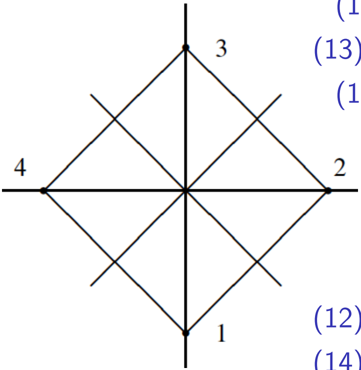

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Question 1

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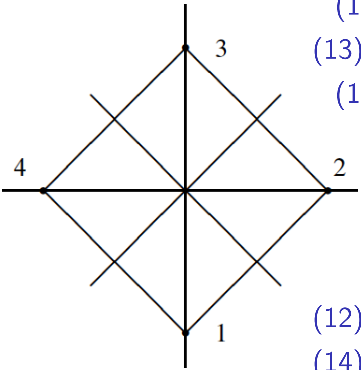

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What is the order of each rigid motion?

Rigid motions of a square: Multiplication table

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	(1)	(1234)	(13)(24)	(1432)	(24)	(12)(34)	(13)	(14)(23)
(1)	(1)	(1234)	(13)(24)	(1432)	(24)	(12)(34)	(13)	(14)(23)
(1234)	(1234)	(13)(24)	(1432)	(1)	(12)(34)	(13)	(14)(23)	(24)
(13)(24)	(13)(24)	(1432)	(1)	(1234)	(13)	(14)(23)	(24)	(12)(34)
(1432)	(1432)	(1)	(1234)	(13)(24)	(14)(23)	(24)	(12)(34)	(13)
(24)	(24)	(14)(23)	(13)	(12)(34)	(1)	(1432)	(13)(24)	(1234)
(12)(34)	(12)(34)	(24)	(14)(23)	(13)	(1234)	(1)	(1432)	(13)(24)
(13)	(13)	(12)(34)	(24)	(14)(23)	(13)(24)	(1234)	(1)	(1432)
(14)(23)	(14)(23)	(13)	(12)(34)	(24)	(1432)	(13)(24)	(1234)	(1)

Example: Rigid motions of an equilateral triangle

Proposition 1

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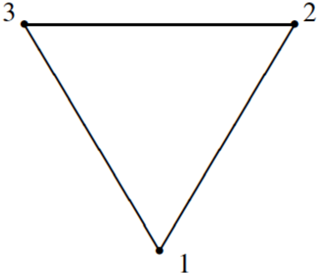
The rigid motions of an equilateral triangle yield the group S_3.



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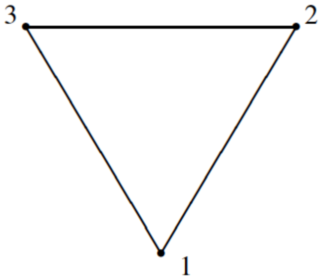

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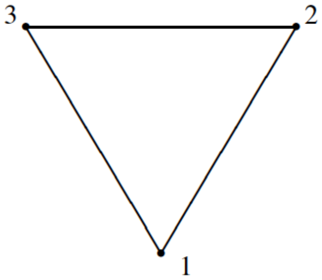

(123) counterclockwise rotation through 120°

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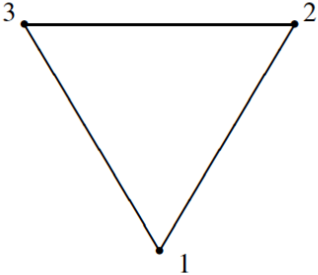

(123) counterclockwise rotation through 120°

(132) counterclockwise rotation through 240°

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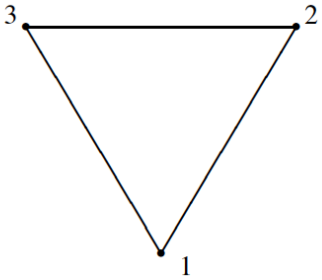

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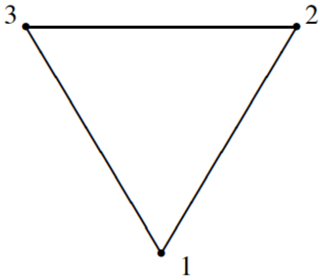

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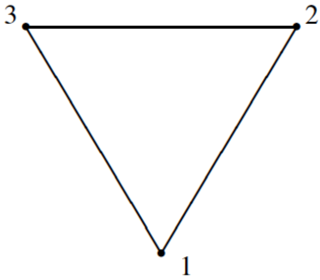

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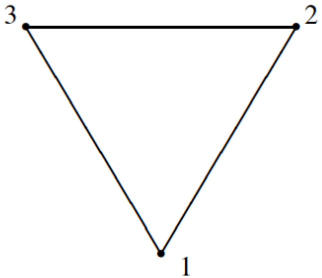

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Note 3 (Another notion for describing S_3 in §3.3)

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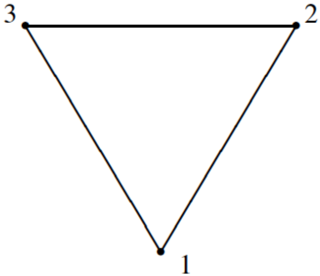

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$S_3 = \{e, a, a^2, b, ab, a^2b\}$, *where* $a^3 = e$, $b^2 = e$, $ba = a^2b = a^{-1}b$.

Note 4 (Another notion for describing Rigid Motions of a Square)

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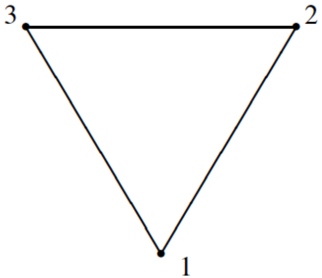

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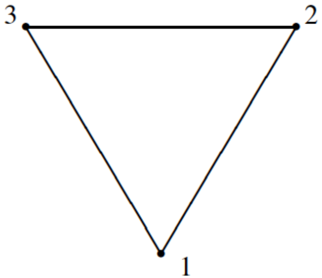

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Note 4 (Another notion for describing Rigid Motions of a Square)

Let $a = (1234)$ *and* $b = (24)$. *It can be shown that* $ba = a^3b$. *The group* $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, *where* $a^4 = e$, $b^2 = e$, $ba = a^3b = a^{-1}b$.

Example: Rigid motions of a regular polygon (n-gon)

Proposition 2

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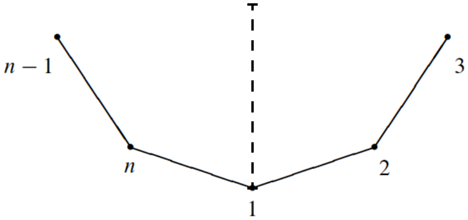

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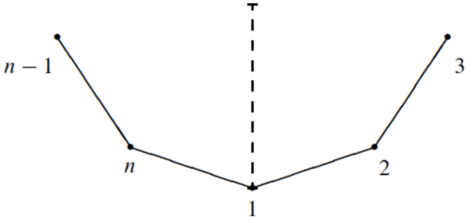

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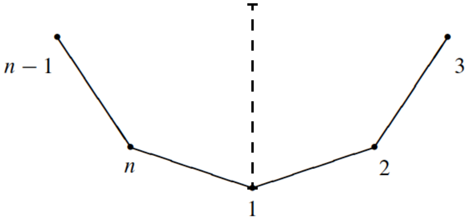

i) Let a be a counterclockwise rotation about the center, through $360/n$ degrees. Thus a is a the cycle $(123 \cdots n)$ of length n and has order n.
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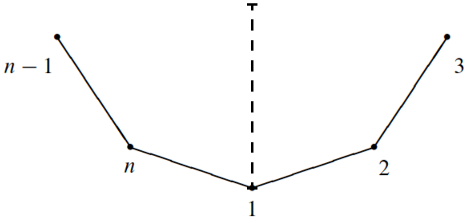

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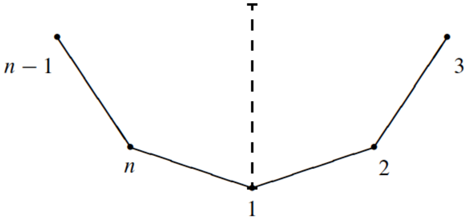

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ii) Let b be a flip about the line of symmetry through position number 1. Thus b has order 2 and is given by the product of transpositions $(2n)(3\ n-1)\cdots$.

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Consider the set $S = \{a^k, a^k b \mid 0 \leq k < n\}$ of rigid motions.

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Thus, $|S| = 2n$, and so $G = S$.

Note 5 (Notion for describing Rigid Motions of a regular n-gon)

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Goal: *To show* $ba = a^{-1} b$.

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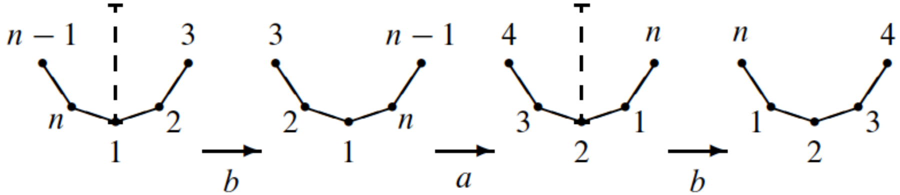

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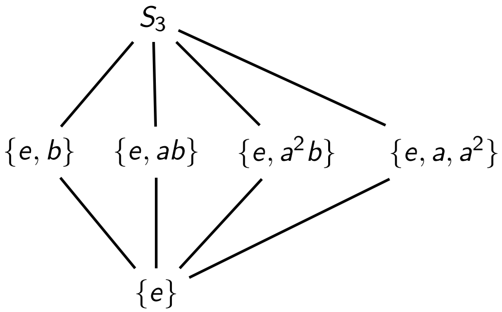

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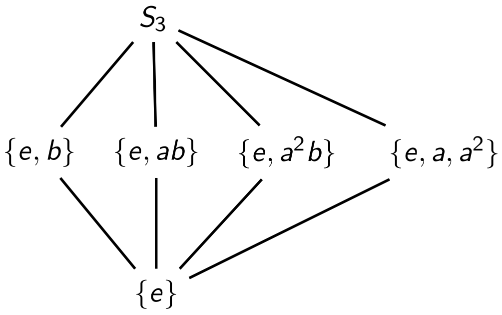

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Note: $D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, where $a^4 = e, b^2 = e, ba = a^3b$.
The possible orders of proper subgroups of D_4 are 1, 2, or 4. (Why?)

I. The trivial subgroups: $\{e\}, D_4$.

II. The cyclic (proper) subgroups:
   (a) a has order 4. In fact, $\langle a \rangle = \langle a^3 \rangle = \{e, a, a^2, a^3\}$. (Why?)
   (b) Each of the elements a^2, b, ab, a^2b, a^3b has order 2. (Check it!)

III. Are there (proper) subgroups of D_4 that are not cyclic? Yes! (How?)
   (i) If H is a non-cyclic (proper) subgroup, then $|H| = 4$. (Why?)
   (ii) Any non-identity element of H has order 2. (Why?)
   (iii) $H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$: Say, $H = \{e, x, y, xy\}$, and so $yx = xy$. (Why?)
   (iv) Consider all possible pairs of elements of order 2 to find all such H's.
   (v) In fact, there are two such groups. (Check it!)
      (1) $H_1 = \{e, a^2, b, a^2b\}$: $ba^2 = (ba)a = a^3(ba) = a^3(a^3b) = a^2b$ ✓
      (2) $H_2 = \{e, a^2, ab, a^3b\}$: $(ab)a^2 = a(ba)a = a(a^3b)a = ba = a^3b$ ✓
   (vi) A cyclic subgroup is the smallest subgroup containing the generator; these subgroups H's are the smallest ones containing the two elements used to construct it.

Subgroups of D_4 cont.: Subgroup diagram

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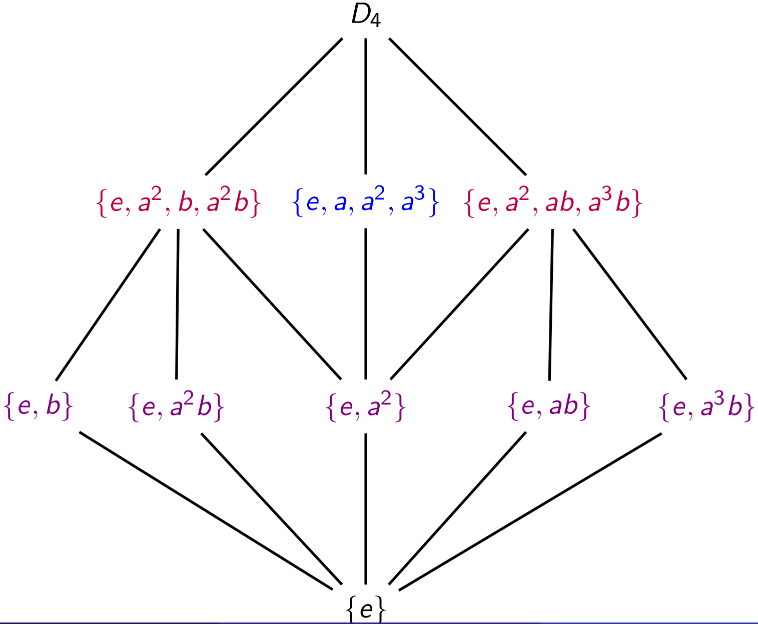

Alternating group A_n

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III. Therefore, $|A_n| = |O_n| = \frac{|S_n|}{2} = \frac{n!}{2}$. (Why?)

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  $$\binom{4}{3} = \text{Four choices:} \qquad 123 \qquad 124 \qquad 134 \qquad 234$$

  For each choice, there are **two** ways to make a cycle. (Why?)
  The following is the list of all cycles of length 3 in S_4:

  $$(123), (132), \qquad (124), (142), \qquad (134), (143), \qquad (234), (243)$$

- two disjoint cycles of length 2: Choose any two of the #s 1, 2, 3, 4

  $$\binom{4}{2} = \text{Six choices:} \qquad 12 \qquad 13 \qquad 14 \qquad 23 \qquad 24 \qquad 34$$

  Each pair of two numbers listed above gives rise to a transposition. The other two numbers form another transposition, which is disjoint from the first one. The order doesn't matter.(Why?) *This implies that there are* **three** *different products of two disjoint transpositions*: Pick any pair of two numbers: 6 choices; the other pair is determined. (Why?) The order doesn't matter $\Rightarrow$ 3 different products. (Why?)

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Standard approach to the definition of even and odd permutations

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Set $X = \{\Delta_n, -\Delta_n\}$. For $\sigma \in S_n$, we define $\widehat{\sigma} : X \to X$ by

$$\widehat{\sigma}(\Delta_n) = \prod_{1 \le i < j \le n} (x_{\sigma(i)} - x_{\sigma(j)}) \text{ and } \widehat{\sigma}(-\Delta_n) = -\prod_{1 \le i < j \le n} (x_{\sigma(i)} - x_{\sigma(j)}).$$

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