§3.3 Constructing Examples

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MATH 546/701I

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 - Corollary 21: Any group of prime order is cyclic (and so abelian).

Any group of order 2, 3, or 5 must be cyclic. (Why?)

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By Fact 1, there is only one possible pattern for the table. (eg. ${\bm Z}_8^\times)$

Table 3.3.1: Multiplication Tables for Groups of Order 4

		а				e	a	b	С
е	е	a	a^2	a^3 e	е	е	a	b	С
a	a	a^2	a^3	е	a	a	е	С	b
a^2	a^2	a^3	e	a	b	b	С	е	a
a^3	a^3	е	а	a^2	С	С	b	a	е
N/:				Contraction Ended				N 4	20.21 2020

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Find its multiplication table. (Check it!)

Multiplication Table for S_3

 $S_3 = \{e, a, a^2, b, ab, a^2b\},$ where $a^3 = e, b^2 = e, ba = a^2b.$ We also calculated $ba^2 = (ba)a = (a^2b)a = a^2(ba) = a^2(a^2b) = ab$. d $ba^2 = (ba)a = (a^2b)a = a^2(ba) = a^2$ $e a a^2 b ab a^2b$ $e e a a^2 b ab a^2b$ $a a^2 e ab a^2b b$ $a^2 a^2 e a a^2b b ab$ $b b a^2b ab e a^2 a$ $ab ab b a^2b a e a^2$ $a^2b a^2b ab b a^2 a e$

Yi

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If the operation of G is denoted additively, then we write H + K, and refer to the **sum** of H and K.

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Let G be a group, and let H and K be subgroups of G. If $h^{-1}kh \in K$ for all $h \in H$ and $k \in K$, then HK is a subgroup of G

Proof.

(i) Closure: Let $g_1, g_2 \in HK$. Then $g_1 = h_1k_1$ and $g_2 = h_2k_2$.

 $g_{1}g_{2} = (h_{1}k_{1})(h_{2}k_{2}) = h_{1}(h_{2}h_{2}^{-1})k_{1}h_{2}k_{2} = h_{1}h_{2}(h_{2}^{-1}k_{1}h_{2})k_{2} \in HK$ We omit parentheses because of associativity. (ii) Identity: $e = e \cdot e \in HK$. (Why?) (iii) Inverses: If g = hk for $h \in H$ and $k \in K$. Then $g^{-1} = k^{-1}h^{-1} = (h^{-1}h)k^{-1}h^{-1} = h^{-1}((h^{-1})^{-1}k^{-1}h^{-1}) \in HK$

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If G is abelian, then the product of any two subgroups is again a subgroup. If G is a finite group, then $|HK| = |H||K|/|H \cap K|$. (How to prove it?) Constructing Examples May 20-21, 2020 7 / 19

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This means that there exists $t \in H \cap K$ such that $t = h'^{-1}h = k'k^{-1}$. So $h' = ht^{-1}$ and k' = tk, i.e., $h'k' = (ht^{-1})(tk)$ for some $t \in H \cap K$.

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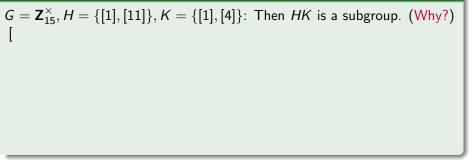
 Thus, every element in HK can be written in exactly |H ∩ K| different ways. Therefore,

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

Example 3

 $G = \mathbf{Z}_{15}^{\times}, H = \{[1], [11]\}, K = \{[1], [4]\}:$

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(a) The direct product G₁ × G₂ is a group under the operation defined for all (a₁, a₂), (b₁, b₂) ∈ G₁ × G₂ by
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Remark 1

If G_1, G_2 are finite groups, then $|G_1 \times G_2| = |G_1| \cdot |G_2|$.

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(iii) Identity:

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(b) Let o(a₁) = n, o(a₂) = m. In G₁ × G₂, o((a₁, a₂)) is the smallest

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$$\begin{aligned} (a_1, a_2)((b_1, b_2)(c_1, c_2)) =& (a_1, a_2)(b_1 * c_1, b_2 \cdot c_2) \\ =& (a_1 * (b_1 * c_1), a_2 \cdot (b_2 \cdot c_2)) \\ =& ((a_1 * b_1) * c_1, (a_2 \cdot b_2) \cdot c_2) \\ =& (a_1 * b_1, a_2 \cdot b_2)(c_1, c_2) \\ =& ((a_1, a_2)(b_1, b_2))(c_1, c_2) \end{aligned}$$

(iii) Identity: (e₁, e₂), where e_i is the identity elements in G_i, i = 1, 2.
(iv) Inverses: (a₁, a₂)⁻¹ = (a₁⁻¹, a₂⁻¹). (Check it!)
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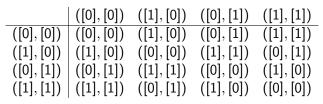
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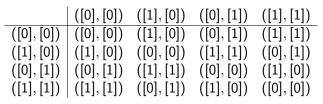
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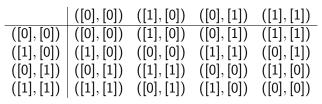
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This group is characterized by the fact that it has order 4 and each element except the identity has order 2.

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$$Z_n \times Z_m$$
 is cyclic if and only if $gcd(n, m) = 1$.

Recall: A finite group G is cyclic if and only if o(x) = |G| for some $x \in G$. \Rightarrow : Assume $\mathbb{Z}_n \times \mathbb{Z}_m$ is cyclic, we need to show (n, m) = 1.

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Therefore, $\mathbf{Z}_n \times \mathbf{Z}_m = \langle ([1]_n, [1]_m) \rangle$ since $|\mathbf{Z}_n \times \mathbf{Z}_m| = o(([1]_n, [1]_m))$.

Definition 8

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(i) the set of all elements of F is an abelian group under +;

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A similar argument shows that $0 \cdot a = 0$ for all $a \in F$. (Check it!)

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Let Q be the following set of matrices in $GL_2(\mathbf{C})$:

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 $ij = k, \quad jk = i, \quad ki = j; \qquad ji = -k, \quad kj = -i, \quad ik = -j.$

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- -1 has order 2
- $\pm i, \pm j$, and $\pm k$ have order 4

Subgroup generated by \boldsymbol{S}

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Proposition 7

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Yi

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(i) If x and y are two words in S, then xy is again a word in S. √
(ii) e = aa⁻¹ ∈ ⟨S⟩. A element a ∈ S always exists since S is nonempty.
(iii) x⁻¹ ∈ ⟨S⟩: reverses the order and changes the sign of the exponent.
If S ⊆ H, where H is a subgroup of G, then it contains all words in S.
Therefore, ⟨S⟩ ⊆ H.

Let S be a nonempty subset of the group G. A finite product of elements of S and their inverses is called a **word** in S. The set of all words in S is denoted by $\langle S \rangle$.

For example, for $a, b, c \in S$, the product $a^{-1}a^{-1}bab^{-1}acb^{-1}cbc^{-1}c^{-1}$.

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Therefore, ⟨S⟩ ⊆ H. It follows that ⟨S⟩ is the intersection of all subgroups of G that contain S. That is, ⟨S⟩ is the smallest subgroup that contains S.