# $\S3.2$ Subgroups

Shaoyun Yi

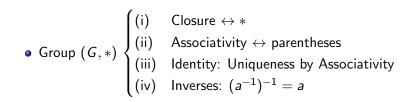
### MATH 546/701I

### University of South Carolina

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## Review



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• Group 
$$(G, *)$$
   

$$\begin{cases}
(i) & \text{Closure} \leftrightarrow * \\
(ii) & \text{Associativity} \leftrightarrow \text{parentheses} \\
(iii) & \text{Identity: Uniqueness by Associativity} \\
(iv) & \text{Inverses: } (a^{-1})^{-1} = a
\end{cases}$$

• Examples:  $(\mathbf{R}^{\times}, \cdot)$ ,  $(\text{Sym}(S), \circ)$ ,  $(M_n(\mathbf{R}), +_{\text{matrix}})$ ,  $(\text{GL}_n(\mathbf{R}), \cdot_{\text{matrix}})$ 

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- Eg. Conjugacy ↔ Equivalence relation

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multiplicative (sub)groups of nonzero elements:  $\{\pm 1\} \subseteq \mathbf{Q}^{\times} \subseteq \mathbf{R}^{\times} \subseteq \mathbf{C}^{\times}$ .

We cannot include the set of nonzero integers in this diagram. (Why?)

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The set of all multiples of a fixed positive integer n, denoted by

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Let G be a group with identity element e, and let H be a subset of G. Then H is a subgroup of G if and only if the following conditions hold:

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(⇐): (i) Closure ✓; (ii) Associativity: If a, b, c ∈ H, then in G we have a(bc) = (ab)c, and so also in H;

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(i)  $ab \in H$  for all  $a, b \in H$ ; (ii)  $e \in H$ ; (iii)  $a^{-1} \in H$  for all  $a \in H$ .

(⇒): (i) is trivial. (Why?);
(ii) Let e' be an identity element for H. To show: e' = e.
e'e' = e' (Why?) and e'e = e' (Why?) ⇒ e'e' = e'e ⇒ e' = e. (Why?)
(iii) If a ∈ H, then a must have an inverse b ∈ H. To show: a<sup>-1</sup> = b. In G, we have ab = e = aa<sup>-1</sup>, and then a<sup>-1</sup> = b. ⇒ a<sup>-1</sup> ∈ H.
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#### Note 1 (To show that the subset H is nonempty:)

The easiest way to do this is to show that H contains the identity element e.

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In the group  $GL_2(\mathbf{R})$ , let H be the following set of matrices:

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The product of any two of these matrices is again in the set H. (Check it!) Since the set is finite and closed under matrix multiplication, Corollary 8 implies that it is a subgroup of  $GL_2(\mathbf{R})$ .

Let *H* be the set of all diagonal matrices in the group  $G = GL_n(\mathbf{R})$ . The diagonal entries of any element in *H* must all be nonzero. (Why?) Let *H* be the set of all diagonal matrices in the group  $G = \operatorname{GL}_n(\mathbf{R})$ . The diagonal entries of any element in *H* must all be nonzero. (Why?) To show that *H* is a subgroup, we can no longer apply Corollary 8. (Why?) Let *H* be the set of all diagonal matrices in the group  $G = \operatorname{GL}_n(\mathbf{R})$ . The diagonal entries of any element in *H* must all be nonzero. (Why?) To show that *H* is a subgroup, we can no longer apply Corollary 8. (Why?) It is probably easiest to just use Proposition 1. Let *H* be the set of all diagonal matrices in the group  $G = \operatorname{GL}_n(\mathbf{R})$ . The diagonal entries of any element in *H* must all be nonzero. (Why?) To show that *H* is a subgroup, we can no longer apply Corollary 8. (Why?) It is probably easiest to just use Proposition 1. (i) If  $a, b \in H$ , then  $ab \in H$ . (Check it!)

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Let G be a group, and let a be any element of G. The set

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 $Z = \langle 1 \rangle = \langle -1 \rangle$ . (Check it!)

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### Revisit Example 9

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#### Remark 1

Every proper subgroup of  $S_3$  is cyclic, but  $S_3$  is not cyclic. Same with  $\mathbf{Z}_8^{\times}$ .

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Furthermore,  $|\langle a \rangle| = o(a)$ .

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Subgroups

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# Lagrange's Theorem

### Theorem 18 (Lagrange)

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Subgroups

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Yi

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Subgroups

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