Some Additional Practice Problems for Final Exam

Review Lecture Slides/Recordings & Homework Assignments

Good luck for the final !

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(1) Find $\gcd(7605, 5733)$, and express it as a linear combination of 7605 and 5733.

(2) Solve the congruence $24x \equiv 168 \pmod{200}$.

(3) Solve the system of congruences $2x \equiv 9 \pmod{15}$ $x \equiv 8 \pmod{11}$.

(4) Let $\sigma = (13579)(126)(1253)$. Find its order and its inverse. Is σ even or odd?

(5) Let $(G, \cdot)$ be a group and let $a \in G$. Define a new operation $*$ on the set G by

$$x * y = x \cdot a \cdot y, \text{ for all } x, y \in G.$$

Show that G is a group under the operation $*$.

(6) For each binary operation $*$ given below, determine whether or not $*$ defines a group structure on the given set. If not, list the group axioms that fail to hold.

(a) Define $*$ on $\mathbf{Z}$ by $a * b = \min\{a, b\}$.

(b) Define $*$ on $\mathbf{Z}^+$ by $a * b = \max\{a, b\}$.

(c) Define $*$ on $\mathbf{Z}$ by $x * y = x^2y^3$.

(d) Define $*$ on $\mathbf{Z}^+$ by $x * y = x^y$.

(e) Define $*$ on $\mathbf{R}$ by $x * y = x + y - 1$.

(f) Define $*$ on $\mathbf{R}^\times$ by $x * y = xy + 1$.

(7) Show that if $|G| = pq$, where $p \neq q$ are prime numbers, then every proper nontrivial subgroup of G is cyclic.

(8) Let K be the following subset of $\mathrm{GL}_2(\mathbf{R})$.

$$K = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{R}) \middle| a = d, c = -2b \right\}$$

Show that K is a subgroup of $\mathrm{GL}_2(\mathbf{R})$.

(9) List all of the generators of the cyclic group $\mathbf{Z}_5 \times \mathbf{Z}_3$.

(10) Find the order of the element $([9]_{12}, [15]_{18})$ in the group $\mathbf{Z}_{12} \times \mathbf{Z}_{18}$.

(11) Show that if $p > 2$ is a prime, then any group of order $2p$ has an element of order 2 and an element of order p.

(12) Prove that

(a) $\mathbf{Z}_{17}^\times \cong \mathbf{Z}_{16}$.

(b) $\mathbf{Z}_{30} \times \mathbf{Z}_2 \cong \mathbf{Z}_{10} \times \mathbf{Z}_6$.

(13) Is $\mathbf{Z}_{20}^\times$ cyclic? Is $\mathbf{Z}_{50}^\times$ cyclic?

(14) (a) In $\mathbf{Z}_{30}$, find the order of the subgroup $\langle [18]_{30} \rangle$; find the order of $\langle [24]_{30} \rangle$.

(b) In $\mathbf{Z}_{45}$, find all elements of order 15.

(15) Prove that if G_1 and G_2 are groups of order 7 and 11, respectively, then the direct product $G_1 \times G_2$ is a cyclic group.

(16) Prove that $D_{12} \not\cong D_4 \times \mathbf{Z}_3$.

(17) For any elements $\sigma, \tau \in S_n$, show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

(18) Find the formulas for all group homomorphisms from $\mathbf{Z}_{18}$ to $\mathbf{Z}_{30}$.

(19) Let G be a group. For $a, b \in G$ we say that b is **conjugate** to a, written $b \sim a$, if there exists $g \in G$ such that $b = gag^{-1}$. Following part (a), the equivalence classes of $\sim$ are called the **conjugacy classes** of G.

(a) Show that $\sim$ is an equivalence relation on G.

(b) Show that $\phi_g : G \to G$ defined by $\phi_g(x) = gxg^{-1}$ is an isomorphism of G.

(c) Show that a subgroup N of the group G is normal in G if and only if N is a union of conjugacy classes.

(20) (a) List the cosets of $\langle [9]_{16} \rangle$ in $\mathbf{Z}_{16}^{\times}$, and find the order of each coset in $\mathbf{Z}_{16}^{\times}/\langle [9]_{16} \rangle$.

(b) List the cosets of $\langle [7]_{16} \rangle$ in $\mathbf{Z}_{16}^{\times}$. Is the factor group $\mathbf{Z}_{16}^{\times}/\langle [7]_{16} \rangle$ cyclic?

(21) Let G be the dihedral group D_6 and let H be the subset $\{e, a^3, b, a^3b\}$ of G.

(a) Show that H is subgroup of G.

(b) Is H a normal subgroup of G?

(22) Let H and N be normal subgroups of a group G, with $N \subseteq H$. Define

$$\phi : G/N \to G/H \text{ by } \phi(xN) = xH, \text{ for all cosets } xN \in G/N.$$

(a) Show that ϕ is a well-defined onto homomorphism.

(b) Show that $(G/N)/(H/N) \cong G/H$.

$* * *$ *The solution is also available on the course website.* $* * *$

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