Homework 9

Due: June 20th (Saturday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.
  (1), (2), (4), (8), (9)

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(1) (a) List all cosets of $\langle [16]_{24} \rangle$ in $\mathbf{Z}_{24}$.

Since $\gcd(16, 24) = 8$, we have $\langle [16]_{24} \rangle = \langle [8]_{24} \rangle = \{[0]_{24}, [8]_{24}, [16]_{24}\}$. Thus,

$$\begin{aligned}
\langle [8]_{24} \rangle =& \{[0]_{24}, [8]_{24}, [16]_{24}\} = \langle [16]_{24} \rangle \\
[1]_{24} + \langle [8]_{24} \rangle =& \{[1]_{24}, [9]_{24}, [17]_{24}\} = [1]_{24} + \langle [16]_{24} \rangle \\
[2]_{24} + \langle [8]_{24} \rangle =& \{[2]_{24}, [10]_{24}, [18]_{24}\} = [2]_{24} + \langle [16]_{24} \rangle \\
[3]_{24} + \langle [8]_{24} \rangle =& \{[3]_{24}, [11]_{24}, [19]_{24}\} = [3]_{24} + \langle [16]_{24} \rangle \\
[4]_{24} + \langle [8]_{24} \rangle =& \{[4]_{24}, [12]_{24}, [20]_{24}\} = [4]_{24} + \langle [16]_{24} \rangle \\
[5]_{24} + \langle [8]_{24} \rangle =& \{[5]_{24}, [13]_{24}, [21]_{24}\} = [5]_{24} + \langle [16]_{24} \rangle \\
[6]_{24} + \langle [8]_{24} \rangle =& \{[6]_{24}, [14]_{24}, [22]_{24}\} = [6]_{24} + \langle [16]_{24} \rangle \\
[7]_{24} + \langle [8]_{24} \rangle =& \{[7]_{24}, [15]_{24}, [23]_{24}\} = [7]_{24} + \langle [16]_{24} \rangle
\end{aligned}$$

(b) List all cosets of $\langle ([1]_3, [2]_6) \rangle$ in $\mathbf{Z}_3 \times \mathbf{Z}_6$.

Since $o(([1]_3, [2]_6))$ in $\mathbf{Z}_3 \times \mathbf{Z}_6$ is $\operatorname{lcm}[3, 3] = 3$, we can calculate that

$$\langle ([1]_3, [2]_6) \rangle = \{([0]_3, [0]_6), ([1]_3, [2]_6), ([2]_3, [4]_6)\}.$$

And so the index is 6, that is, there are six cosets of $\langle ([1]_3, [2]_6) \rangle$ in $\mathbf{Z}_3 \times \mathbf{Z}_6$:

$$\begin{aligned}
\langle ([1]_3, [2]_6) \rangle =& \{([0]_3, [0]_6), ([1]_3, [2]_6), ([2]_3, [4]_6)\} \\
([0]_3, [1]_6) + \langle ([1]_3, [2]_6) \rangle =& \{([0]_3, [1]_6), ([1]_3, [3]_6), ([2]_3, [5]_6)\} \\
([0]_3, [2]_6) + \langle ([1]_3, [2]_6) \rangle =& \{([0]_3, [2]_6), ([1]_3, [4]_6), ([2]_3, [0]_6)\} \\
([0]_3, [3]_6) + \langle ([1]_3, [2]_6) \rangle =& \{([0]_3, [3]_6), ([1]_3, [5]_6), ([2]_3, [1]_6)\} \\
([0]_3, [4]_6) + \langle ([1]_3, [2]_6) \rangle =& \{([0]_3, [4]_6), ([1]_3, [0]_6), ([2]_3, [2]_6)\} \\
([0]_3, [5]_6) + \langle ([1]_3, [2]_6) \rangle =& \{([0]_3, [5]_6), ([1]_3, [1]_6), ([2]_3, [3]_6)\}
\end{aligned}$$

(2) For each of the subgroups $\{e, a^2\}$ and $\{e, b\}$ of D_4, list all left and right cosets.

$D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, where $a^4 = e, b^2 = 2$, and $ba = a^{-1}b = a^3b$.

left cosets		right cosets	
	$\{e, a^2\}$	$\{e, a^2\}$	
$a\{e, a^2\} =$	$\{a, a^3\}$	$\{a, a^3\}$	$= \{e, a^2\}a$
$b\{e, a^2\} =$	$\{b, a^2b\}$	$\{b, a^2b\}$	$= \{e, a^2\}b$
$ab\{e, a^2\} =$	$\{ab, a^3b\}$	$\{ab, a^3b\}$	$= \{e, a^2\}ab$

left cosets		right cosets	
	$\{e, b\}$	$\{e, b\}$	
$a\{e, b\} =$	$\{a, ab\}$	$\{a, a^3b\}$	$= \{e, b\}a$
$a^2\{e, b\} =$	$\{a^2, a^2b\}$	$\{a^2, a^2b\}$	$= \{e, b\}a^2$
$a^3\{e, b\} =$	$\{a^3, a^3b\}$	$\{a^3, ab\}$	$= \{e, b\}a^3$

(3) Prove that if N is a normal subgroup of G, and H is any subgroup of G, then $H \cap N$ is a normal subgroup of H.

Proof. Since we already know that $H \cap N$ is always a subgroup, we only need to prove that it is normal. For any $a \in H \cap N$ and any $h \in H$, we have

$hah^{-1} \in H$ since $a \in H, h \in H$, and H is a subgroup.

$hah^{-1} \in N$ since $a \in N$, and N is a normal subgroup of G.

This implies $hah^{-1} \in H \cap N$, and so $H \cap N$ is a normal subgroup of H. □

(4) Let N be a normal subgroup of index m in G. Show that $a^m \in N$ for all $a \in G$.

Proof. By $[G : N] = m$, we know that the factor group G/N has order m. Then

$$(aN)^m = N \text{ for all } a \in G.$$

It follows from the coset multiplication that $(aN)^m = a^m N = N$, i.e., $a^m \in N$. □

(5) Let N be a normal subgroup of G. Show that the order of any coset aN in G/N is a divisor of $o(a)$, when $o(a)$ is finite.

Proof. Let $o(a) = n$. Then we have $a^n = e$. We also see that

$$(aN)^n = a^n N = eN = N.$$

Then the order of aN in G/N is a divisor of n. □

(6) Let H and K be normal subgroups of G such that $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Proof. It suffices to show $k^{-1}hkh^{-1} = e$ for all $h \in H$ and $k \in K$. In particular,

$k^{-1}hkh^{-1} = (k^{-1}hk)h^{-1} \in H$ since H is a normal subgroup of G.

$k^{-1}hkh^{-1} = k^{-1}(hkh^{-1}) \in K$ since K is a normal subgroup of G.

This implies that $k^{-1}hkh^{-1} \in H \cap K = \{e\}$. This completes the proof. □

(7) If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G. (*Note that you need to show that NM is a subgroup of G first.*)

Proof. We are going to separate two steps to prove.

Claim 0.1. *NM is a subgroup of G.*

Proof of Claim 0.1. We are going to use Corollary 7 in Section 3.2 (Lecture slides). NM is nonempty since $e = ee \in NM$. For any $n_1m_1, n_2m_2 \in NM$ with $n_1, n_2 \in N$ and $m_1, m_2 \in M$, we need to show that $(n_1m_1)(n_2m_2)^{-1} \in NM$. In fact,

$$\begin{aligned}(n_1m_1)(n_2m_2)^{-1} =&(n_1m_1)(m_2^{-1}n_2^{-1})\\ =&n_1(m_1m_2^{-1})n_2^{-1}\\ =&(n_1n_2^{-1}) \cdot n_2(m_1m_2^{-1})n_2^{-1} \overset{!}{\in} NM\end{aligned}$$

$\overset{!}{\in}$ holds since N is a subgroup of G and M is a normal subgroup of G. □

Claim 0.2. *NM is normal.*

Proof of Claim 0.2. For any $nm \in NM$ and any $g \in G$, we have

$$gnmg^{-1} = (gng^{-1})(gmg^{-1}) \in NM$$

because N and M are normal subgroups of G. Hence proved. □

□

(8) Compute the factor group $(\mathbf{Z}_6 \times \mathbf{Z}_4)/\langle([2]_6, [2]_4)\rangle$.

First, we observe that $o(([2]_6, [2]_4))$ in $\mathbf{Z}_6 \times \mathbf{Z}_4$ is $\text{lcm}[3, 2] = 6$. In particular,

$$\langle([2]_6, [2]_4)\rangle = \{([0]_6, [0]_4), ([2]_6, [2]_4), ([4]_6, [0]_4), ([0]_6, [2]_4), ([2]_6, [0]_4), ([4]_6, [2]_4)\}.$$

There are four elements in the factor group $(\mathbf{Z}_6 \times \mathbf{Z}_4)/\langle([2]_6, [2]_4)\rangle$:

$$\begin{aligned}\langle([2]_6, [2]_4)\rangle =&\{([0]_6, [0]_4), ([2]_6, [2]_4), ([4]_6, [0]_4), ([0]_6, [2]_4), ([2]_6, [0]_4), ([4]_6, [2]_4)\}\\ ([0]_6, [1]_4) + \langle([2]_6, [2]_4)\rangle =&\{([0]_6, [1]_4), ([2]_6, [3]_4), ([4]_6, [1]_4), ([0]_6, [3]_4), ([2]_6, [1]_4), ([4]_6, [3]_4)\}\\ ([1]_6, [0]_4) + \langle([2]_6, [2]_4)\rangle =&\{([1]_6, [0]_4), ([3]_6, [2]_4), ([5]_6, [0]_4), ([1]_6, [2]_4), ([3]_6, [0]_4), ([5]_6, [2]_4)\}\\ ([1]_6, [1]_4) + \langle([2]_6, [2]_4)\rangle =&\{([1]_6, [1]_4), ([3]_6, [3]_4), ([5]_6, [1]_4), ([1]_6, [3]_4), ([3]_6, [1]_4), ([5]_6, [3]_4)\}\end{aligned}$$

Let $G = \mathbf{Z}_6 \times \mathbf{Z}_4$ and let $N = \langle([2]_6, [2]_4)\rangle$. Then the factor group G/N is

$$G/N = \{N, ([0]_6, [1]_4) + N, ([1]_6, [0]_4) + N, ([1]_6, [1]_4) + N\}.$$

It is clear that all the non-identity elements in G/N have order 2. In particular,

$$\begin{aligned}2([0]_6, [1]_4) &= ([0]_6, [2]_4) \in N\\ 2([1]_6, [0]_4) &= ([2]_6, [0]_4) \in N\\ 2([1]_6, [1]_4) &= ([2]_6, [2]_4) \in N\end{aligned}$$

In conclusion, the factor group $(\mathbf{Z}_6 \times \mathbf{Z}_4)/\langle([2]_6, [2]_4)\rangle$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

(9) Show that $\mathbf{R}^\times/\langle -1\rangle$ is isomorphic to the group of positive real numbers under multiplication.

Proof. Define $\phi : \mathbf{R}^\times \to \mathbf{R}^+$ by $\phi(x) = |x|$.

(i) ϕ is well-defined: Trivial since $|x| > 0$ for all $x \in \mathbf{R}^\times$.

(ii) ϕ is a homomorphism: For all $x, y \in \mathbf{R}^\times$, we have

$$\phi(xy) = |xy| = |x||y| = \phi(x)\phi(y).$$

(iii) ϕ is onto: For any $x \in \mathbf{R}^+$, we have $\phi(x) = |x| = x$ since $x > 0$.

(iv) $\ker(\phi) = \{x \in \mathbf{R}^\times \mid \phi(x) = |x| = 1\} = \{x \mid x = \pm 1\} = \{1, -1\} = \langle -1 \rangle$.

The desired results follow from the fundamental homomorphism theorem, i.e.,

$$\mathbf{R}^\times / \langle -1 \rangle \cong \mathbf{R}^+.$$

□

(10) Let H and N be subgroups of a group G, and assume that N is a normal subgroup of G. Prove the following statements.

(a) N is a normal subgroup of HN.

Proof. Note that the proof of "HN is a subgroup" is similar as in Question (7). So we just show that it is normal. For any $a \in N$ and any $g = hn \in HN$ with $h \in H$ and $n \in N$, we have

$$gag^{-1} = (hn)a(hn)^{-1} = h(nan^{-1})h^{-1} \overset{!}{\in} N.$$

$\overset{!}{\in}$ holds since N is a normal subgroup of G. This completes the proof. □

(b) Each element of HN/N has the form hN, for some $h \in H$.

Proof. Each element of HN/N has the form hnN with $h \in H$ and $n \in N$, and so we have $hnN = hN$ since $nN = N$. Hence proved. □

(c) Define $\phi : H \to HN/N$ by $\phi(h) = hN$, for all $h \in H$, is an onto homomorphism.

Proof.

(i) ϕ is well-defined: It is true because of part (b).

(ii) ϕ is a homomorphism: For any $h_1, h_2 \in H$, we have

$$\phi(h_1h_2) = h_1h_2N \overset{!}{=} h_1Nh_2N = \phi(h_1)\phi(h_2).$$

$\overset{!}{\in}$ holds since N is a normal subgroup of G.

(iii) ϕ is onto: It is trivial by the definition of ϕ.

□

(d) $HN/N \cong H/K$, where $K = H \cap N$.

Proof. $\ker(\phi) = \{h \in H \mid \phi(h) = hN = N\} = \{h \in H \mid h \in N\} = H \cap N$. It follows from the fundamental homomorphism theorem that

$$HN/N \cong H/H \cap N.$$

□