Homework 8

Due: June 15th (Monday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded. (1), (2), (3), (5), (6)

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(1) Write down the formulas for all homomorphisms from $\mathbf{Z}_{24}$ into $\mathbf{Z}_{18}$.

Define $\phi : \mathbf{Z}_{24} \to \mathbf{Z}_{18}$ by $\phi([x]_{24}) = [mx]_{18}$ for some $[m]_{18} \in \mathbf{Z}_{18}$. In order for ϕ to be well-defined, we need the condition that $18|24m$. That is, $3|4m$, and so $3|m$ since $\gcd(3,4) = 1$. Then, all the possible $[m]_{18}$'s are $[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}$ and $[15]_{18}$. Thus, the formulas for all homomorphisms from $\mathbf{Z}_{24}$ into $\mathbf{Z}_{18}$ are:

$$\begin{aligned}
\phi_0([x]_{24}) =& [0]_{18}\\
\phi_3([x]_{24}) =& [3x]_{18}\\
\phi_6([x]_{24}) =& [6x]_{18}\\
\phi_9([x]_{24}) =& [9x]_{18}\\
\phi_{12}([x]_{24}) =& [12x]_{18}\\
\phi_{15}([x]_{24}) =& [15x]_{18}
\end{aligned}$$

defined for all $[x]_{24} \in \mathbf{Z}_{24}$.

(2) Write down the formulas for all homomorphisms from $\mathbf{Z}$ onto $\mathbf{Z}_{12}$.

Every homomorphism $\phi : \mathbf{Z} \to \mathbf{Z}_{12}$ is defined by $\phi(x) = [mx]_{12}$ for $[m]_{12} \in \mathbf{Z}_{12}$. Moreover, the homomorphism ϕ is onto. This implies that ϕ sends the generator 1 in $\mathbf{Z}$ to the generator $[m]_{12}$ in $\mathbf{Z}_{12}$. As we know that $[m]_{12}$ generates $\mathbf{Z}_{12}$ if and only if $[m]_{12} \in \mathbf{Z}_{12}^{\times}$, i.e., $\gcd(m, 12) = 1$. Thus, $m = 1, 5, 7, 11$. In conclusion, the formulas for all homomorphisms from $\mathbf{Z}$ onto $\mathbf{Z}_{12}$ are:

$$\begin{aligned}
\phi_1(x) =& [x]_{12}\\
\phi_5(x) =& [5x]_{12}\\
\phi_7(x) =& [7x]_{12}\\
\phi_{11}(x) =& [11x]_{12}
\end{aligned}$$

defined for all $x \in \mathbf{Z}$.

(3) For the group homomorphism $\phi : \mathbf{Z}_{15}^{\times} \to \mathbf{Z}_{15}^{\times}$ defined by $\phi([x]) = [x]^2$ for all $[x] \in \mathbf{Z}_{15}^{\times}$, find the kernel and image of ϕ.

Note that $\mathbf{Z}_{15}^{\times} = \{[1], [2], [4], [7], [8], [11], [13], [14]\}$.

$[x]$	$[1]$	$[2]$	$[4]$	$[7]$	$[8]$	$[11]$	$[13]$	$[14]$
$\phi([x]) = [x]^2$	$[1]$	$[4]$	$[1]$	$[4]$	$[4]$	$[1]$	$[4]$	$[1]$

Thus, $\ker(\phi) = \{[1], [4], [11], [14]\}$ and $\text{im}(\phi) = \{[1], [4]\}$.

(4) Define $\phi : \mathbf{C}^\times \to \mathbf{R}^\times$ by $\phi(a + bi) = a^2 + b^2$, for all $a + bi \in \mathbf{C}^\times$. Show that ϕ is a homomorphism.

The well-definedness of ϕ is trivial. For any $a + bi, c + di \in \mathbf{C}^\times$, we have

$$\begin{aligned}\phi((a+bi)(c+di)) =& \phi((ac-bd) + (ad+bc)i) \\ =& (ac-bd)^2 + (ad+bc)^2 \\ =& a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 \\ =& (a^2+b^2)(c^2+d^2) \\ =& \phi((a+bi)) \cdot \phi((c+di))\end{aligned}$$

Thus, ϕ is a homomorphism.

(5) Which of the following functions are homomorphisms? You need to show work to support your answers.

(a) $\phi : \mathbf{R}^\times \to \text{GL}_2(\mathbf{R})$ defined by $\phi(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

Yes:

(i) well-defined: $\phi(a) \in \text{GL}_2(\mathbf{R})$ since $a \in \mathbf{R}^\times$.

(ii) For any $a, b \in \mathbf{R}^\times$, we have

$$\begin{aligned}\phi(a \cdot b) =& \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix} \\ =& \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \\ =& \phi(a)\phi(b)\end{aligned}$$

(b) $\phi : \mathbf{R} \to \text{GL}_2(\mathbf{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

Yes:

(i) well-defined: $\phi(a) \in \text{GL}_2(\mathbf{R})$ since $\det(\phi(a)) = 1 \neq 0$ for all $a \in \mathbf{R}$.

(ii) For any $a, b \in \mathbf{R}^\times$, we have

$$\begin{aligned}\phi(a + b) =& \begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix} \\ =& \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \\ =& \phi(a)\phi(b)\end{aligned}$$

(c) $\phi : \text{M}_2(\mathbf{R}) \to \mathbf{R}$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a$

Yes:

(i) well-defined: Trivial.

(ii) For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \mathrm{M}_2(\mathbf{R})$, we have

$$\begin{aligned} \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = & \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \\ =& a + a' \\ =& \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \phi\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) \end{aligned}$$

(d) $\phi : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{R}^\times$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$

No: ϕ is *not* well-defined. For example, let $b = 0$, and so $ab = 0 \notin \mathbf{R}^\times$.

(6) Let $\phi : G_1 \to G_2$ and $\theta : G_2 \to G_3$ be group homomorphisms. Prove that

(a) $\theta\phi : G_1 \to G_3$ is a homomorphism.

(i) well-defined: For any $a \in G_1$, $\theta\phi(a) = \theta(\phi(a)) \in G_3$ since $\phi(a) \in G_2$.

(ii) For any $a, b \in G_1$, we have

$$\theta\phi(a * b) = \theta(\phi(a * b)) = \theta(\phi(a) \cdot \phi(b)) = \theta(\phi(a)) \star \theta(\phi(b)) = \theta\phi(a) \star \theta\phi(b).$$

(b) $\ker(\phi) \subseteq \ker(\theta\phi)$.

For any $a \in \ker(\phi)$, we have $\theta\phi(a) = \theta(\phi(a)) = \theta(e_2) = e_3$, and so $a \in \ker(\theta\phi)$. This proves $\ker(\phi) \subseteq \ker(\theta\phi)$.

(7) Let G be a group, and let H be a normal subgroup of G. Show that for each $g \in G$ and $h \in H$ there exist h_1 and h_2 in H with $gh = h_1 g$ and $hg = gh_2$.

By definition of the normal subgroup, for each $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$. Say, $ghg^{-1} = h_1$, and so $gh = h_1 g$. Since G is a group, $g^{-1} \in G$. Then $g^{-1}h(g^{-1})^{-1} = g^{-1}hg \in H$. Say $g^{-1}hg = h_2$, and so $hg = gh_2$.

(8) Recall that the center $Z(G)$ of a group G is

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}.$$

Prove that the center of any group is a normal subgroup.

For any $a \in Z(G)$ (it is already a subgroup of G) and any $g \in G$, we have

$$gag^{-1} = gg^{-1}a = ea = a \in Z(G).$$

Thus, the center of any group is a normal subgroup.

(9) Prove that the intersection of two normal subgroups is a normal subgroup.

Let H_1 and H_2 be two normal subgroups of G. Let $H = H_1 \cap H_2$. It is also easy to see that H is a subgroup of G. It suffices to show that H is normal. Let h be any element in H and g be any element in G. Then we have

$$ghg^{-1} \in H_1 \text{ since } h \in H_1 \text{ and } H_1 \text{ is a normal subgroup of } G;$$
$$ghg^{-1} \in H_2 \text{ since } h \in H_2 \text{ and } H_2 \text{ is a normal subgroup of } G.$$

This implies that $ghg^{-1} \in H_1 \cap H_2 = H$. Thus, $H = H_1 \cap H_2$ is again a normal subgroup of G.