Homework 8

Due: June 15th (Monday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.

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(1) Write down the formulas for all homomorphisms from $\mathbf{Z}_{24}$ into $\mathbf{Z}_{18}$.

(2) Write down the formulas for all homomorphisms from $\mathbf{Z}$ onto $\mathbf{Z}_{12}$.

(3) For the group homomorphism $\phi : \mathbf{Z}_{15}^{\times} \to \mathbf{Z}_{15}^{\times}$ defined by $\phi([x]) = [x]^2$ for all $[x] \in \mathbf{Z}_{15}^{\times}$, find the kernel and image of ϕ.

Note that $\mathbf{Z}_{15}^{\times} = \{[1], [2], [4], [7], [8], [11], [13], [14]\}$.

(4) Define $\phi : \mathbf{C}^{\times} \to \mathbf{R}^{\times}$ by $\phi(a + bi) = a^2 + b^2$, for all $a + bi \in \mathbf{C}^{\times}$. Show that ϕ is a homomorphism.

(5) Which of the following functions are homomorphisms? You need to show work to support your answers.

(a) $\phi : \mathbf{R}^{\times} \to \mathrm{GL}_2(\mathbf{R})$ defined by $\phi(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\phi : \mathbf{R} \to \mathrm{GL}_2(\mathbf{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$

(c) $\phi : \mathrm{M}_2(\mathbf{R}) \to \mathbf{R}$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a$

(d) $\phi : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{R}^{\times}$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$

(6) Let $\phi : G_1 \to G_2$ and $\theta : G_2 \to G_3$ be group homomorphisms. Prove that

(a) $\theta\phi : G_1 \to G_3$ is a homomorphism.

(b) $\ker(\phi) \subseteq \ker(\theta\phi)$.

(7) Let G be a group, and let H be a normal subgroup of G. Show that for each $g \in G$ and $h \in H$ there exist h_1 and h_2 in H with $gh = h_1 g$ and $hg = gh_2$.

(8) Recall that the center $Z(G)$ of a group G is

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}.$$

Prove that the center of any group is a normal subgroup.

(9) Prove that the intersection of two normal subgroups is a normal subgroup.