Homework 7

Due: June 8th (Monday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.
  (1), (2), (4), (5), (7)

(1) Find the orders of each of these permutations.

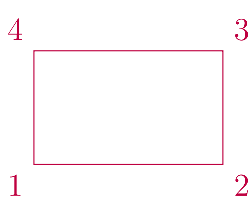
(a) (123)(2435)(132)

= (1534), so order is 4.

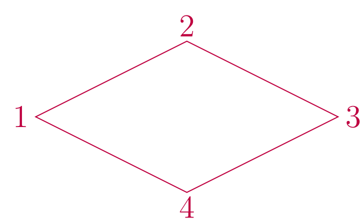
(b) (136)(278)(42537)

= (138256)(47), so order is $\text{lcm}[6, 2] = 6$.

(2) Find the permutations that correspond to the rigid motions of a rectangle that is not a square. Do the same for the rigid motions of a rhombus (diamond) that is not a square.



Rigid motions of the rectangle:
(1), (14)(23), (12)(34), (13)(24).



Rigid motions of the rhombus:
(1), (13), (24), (13)(24).

(3) Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$.

(a) Show that $ba^m = a^{-m}b$, for all $m \in \mathbf{Z}$.

For any positive integer m, we have
$ba^m = (ba)a^{m-1} = (a^{-1}b)a^{m-1} = a^{-1}(ba)a^{m-2} = a^{-2}(ba^{m-2}) = \cdots = a^{-m}b.$
If m is a negative integer, then $m = -|m|$, and so we have

$$ba^m = ba^{-|m|} = b(a^{-1})^{|m|} = (a^{-1})^{-|m|}b = (a^{-1})^m b = a^{-m}b.$$

It is trivial for $m = 0$. In conclusion, we have $ba^m = a^{-m}b$, for all $m \in \mathbf{Z}$.

(b) Show that $ba^m b = a^{-m}$, for all $m \in \mathbf{Z}$.

By (a), we have $ba^m = a^{-m}b$. Thus, $ba^m b = a^{-m}bb = a^{-m}$ for all $m \in \mathbf{Z}$.

(4) Find the order of each element of D_6.

We know that
$D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$, where $a^6 = e, b^2 = e, ba = a^{-1}b$.

By Proposition 1 in §3.5, we know that $o(a^j) = \dfrac{n}{\gcd(j,n)}$. Thus,

	e	a	a^2	a^3	a^4	a^5
order	1	6	3	2	3	6

Claim: All the remaining elements of the form $a^j b$ have the order 2 for $0 \le j < 6$.

$$(a^j b)^2 = a^j (b a^j) b \overset{!}{=} a^j (a^{-j} b) b = (a^j a^{-j})(bb) = ee = e.$$

$\overset{!}{=}$ holds because of Question 3 (a). Note that the above claim holds for any D_n.

(5) Let $\tau = (abc)$ and let σ be any permutation. Show that $\sigma\tau\sigma^{-1} = (\sigma(a)\sigma(b)\sigma(c))$.

$\sigma\tau\sigma^{-1}(\sigma(a)) = \sigma(\tau(\sigma^{-1}\sigma(a))) = \sigma(\tau(a)) = \sigma(b)$: This implies $\sigma(a) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(b)$. Similarly, we can check that $\sigma(b) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(c)$ and $\sigma(c) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(a)$. □

(6) In general, if $(12\cdots k)$ is a cycle of length k and σ is any permutation, then

$$\sigma(12\cdots k)\sigma^{-1} = (\sigma(1)\sigma(2)\cdots\sigma(k)).$$

This is just a generalization of Question (5). For any $1 \le i < k$, we have

$$\sigma(12\cdots k)\sigma^{-1}(\sigma(i)) = \sigma((12\cdots k)(\sigma^{-1}\sigma(i))) = \sigma((12\cdots k)(i)) = \sigma(i+1).$$

And for $i = k$, we just take $i+1$ as 1. □

(7) (a) In S_4, find the subgroup H generated by (123) and (23).

Since $(123) \in H$, $\langle(123)\rangle = \{(1),(123),(132)\} \subseteq H$.
Since $(23) \in H$, $\langle(23)\rangle = \{(1),(23)\} \subseteq H$.
Since H is a group, so by the closure axiom H contains

$$\{(1),(123),(132),(23),(123)(23) = (12),(132)(23) = (13)\}.$$

And this set is closed under multiplication. In fact, $H \cong S_3$. Thus,

$$H = \{(1),(123),(132),(23),(12),(13)\}.$$

(b) For $\sigma = (234)$, find the corresponding subgroup $\sigma H \sigma^{-1}$.

We need to compute $\sigma\tau\sigma^{-1}$ for each $\tau \in H$. First, $\sigma(1)\sigma^{-1} = \sigma\sigma^{-1} = (1)$. The calculations will be easier if we apply Question (5) or (6). In particular,

$$\begin{aligned}
\sigma(123)\sigma^{-1} &= (\sigma(1)\sigma(2)\sigma(3)) = (134)\\
\sigma(132)\sigma^{-1} &= (\sigma(1)\sigma(3)\sigma(2)) = (143)\\
\sigma(23)\sigma^{-1} &= (\sigma(2)\sigma(3)) = (34)\\
\sigma(12)\sigma^{-1} &= (\sigma(1)\sigma(2)) = (13)\\
\sigma(13)\sigma^{-1} &= (\sigma(1)\sigma(3)) = (14)
\end{aligned}$$

Thus, $\sigma H \sigma^{-1} = \{(1),(134),(143),(34),(13),(14)\}$.

(8) Let permutations in S_4 act on polynomials in four variables by permuting the subscripts, as in Theorem 10 in §3.6.

(a) Which permutations in S_4 leave the polynomial $(x_1 - x_2)(x_3 - x_4)$ unchanged?

(i) If x_1 does not change, neither does x_2. It follows that x_3 and x_4 won't change either. The corresponding permutation is (1).

(ii) If $x_1 \mapsto x_2$, then $x_2 \mapsto x_1$. It follows that $x_3 \mapsto x_4$ and $x_4 \mapsto x_3$. So the corresponding permutation is (12)(34).

(iii) If $x_1 \mapsto x_3$, then $x_2 \mapsto x_4$. It follows that $x_3 \mapsto x_1$ and $x_4 \mapsto x_2$. So the corresponding permutation is (13)(24).

(iv) If $x_1 \mapsto x_4$, then $x_2 \mapsto x_3$. It follows that $x_3 \mapsto x_2$ and $x_4 \mapsto x_1$. So the corresponding permutation is (14)(23).

(b) Which permutations in S_4 leave the polynomial $\prod_{1 \leq i < j \leq 4} (x_i + x_j)$ unchanged?

The polynomial $\prod_{1 \leq i < j \leq 4} (x_i + x_j)$ is a symmetric polynomial. All the elements in S_4 leave it unchanged. In particular, let $1 \leq i < j \leq 4$. For any permutation σ in S_4, if $\sigma(i) < \sigma(j)$, then $x_{\sigma(i)} + x_{\sigma(j)}$ is one of the factors, and if $\sigma(i) > \sigma(j)$ then $x_{\sigma(i)} + x_{\sigma(j)} = x_{\sigma(j)} + x_{\sigma(i)}$ is again one factor in this polynomial. Thus, σ leaves the polynomial unchanged.