Homework 6

Due: June 5th (Friday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.

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(1) Finish the proof of Lemma 13 in Lecture Slides_§3.5.

(2) Let G be a group and let $a \in G$ be an element of order 30. List the powers of a that have order 2, order 3 or order 5.

(3) Give the subgroup diagrams of the following groups.
  (a) $\mathbf{Z}_{24}$
  (b) $\mathbf{Z}_{36}$

(4) Which of $\mathbf{Z}_{18}^{\times}, \mathbf{Z}_{20}^{\times}$ are cyclic? (*Do not use Primitive Root Theorem.*)

(5) Find all cyclic subgroups of $\mathbf{Z}_6 \times \mathbf{Z}_3$.

(6) Prove that $\mathbf{Z}_{10}^{\times}$ is not isomorphic to $\mathbf{Z}_{12}^{\times}$. (*Do not use Primitive Root Theorem.*)

(7) You need to show work to support your conclusions.
  (a) Is $\mathbf{Z}_3 \times \mathbf{Z}_{30}$ isomorphic to $\mathbf{Z}_6 \times \mathbf{Z}_{15}$?
  (b) Is $\mathbf{Z}_9 \times \mathbf{Z}_{14}$ isomorphic to $\mathbf{Z}_6 \times \mathbf{Z}_{21}$?

(8) Prove that any cyclic group with more than two elements has at least two different generators.

(9) Prove that any finite cyclic group with more than two elements has an even number of distinct generators.

(10) Let G be the set of all 3×3 matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$.
  (a) Show that if $a, b, c \in \mathbf{Z}_3$, then G is a group with exponent 3.
  (b) Show that if $a, b, c \in \mathbf{Z}_2$, then G is a group with exponent 4.

(11) Let G be any group with no proper, nontrivial subgroups, and assume that $|G| > 1$. Prove that G must be isomorphic to $\mathbf{Z}_p$ for some prime p.