Homework 4

Due: May 25th (Monday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.

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(1) Find HK in $\mathbf{Z}_{16}^{\times}$, if $H = \langle [3] \rangle$ and $K = \langle [5] \rangle$.

(2) Find the order of the element $([9]_{12}, [15]_{18})$ in the group $\mathbf{Z}_{12} \times \mathbf{Z}_{18}$.

(3) Prove that if G_1 and G_2 are abelian groups, then the direct product $G_1 \times G_2$ is abelian.

(4) Construct an abelian group of order 12 that is not cyclic.

(5) Construct a group of order 12 that is not abelian.

(6) Let G_1 and G_2 be groups, with subgroups H_1 and H_2, respectively. Show that

$$\{(x_1, x_2) \mid x_1 \in H_1, x_2 \in H_2\}$$

is a subgroup of the direct product $G_1 \times G_2$.

(7) (a) Let $C_1 = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a = b\}$. Show that C_1 is a subgroup of $\mathbf{Z} \times \mathbf{Z}$.

(b) For each positive integer $n \geq 2$, let $C_n = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \equiv b \pmod{n}\}$. Show that C_n is a subgroup of $\mathbf{Z} \times \mathbf{Z}$.

(c) Show that every proper subgroup of $\mathbf{Z} \times \mathbf{Z}$ that contains C_1 has the form C_n, for some positive integer n.

(8) Let G_1 and G_2 be groups, and let G be the direct product $G_1 \times G_2$. Let $H = \{(x_1, x_2) \in G_1 \times G_2 \mid x_2 = e_2\}$ and let $K = \{(x_1, x_2) \in G_1 \times G_2 \mid x_1 = e_1\}$.

(a) Show that H and K are subgroups of G.

(b) Show that $HK = KH = G$.

(c) Show that $H \cap K = \{(e_1, e_2)\}$.

(9) Let H, K, L be subgroups of the group G, with $H \subseteq K$. Prove that

$$H(K \cap L) = K \cap HL.$$

Note: This is an equality of sets, since they may not be subgroups.

(10) Let F be a field, and let H be the subset of $\mathrm{GL}_2(F)$ consisting of all invertible upper triangular matrices. Show that H is a subgroup of $\mathrm{GL}_2(F)$.