Homework 2

Due: May 18th (Monday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.

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(1) Using ordinary addition of integers as the operation, show that the set of even integers is a group, but that the set of odd integers is not.

(2) For each binary operation $*$ defined on a set below, determine whether or not $*$ gives a group structure on the set. If it is **not** a group, **say which axioms fail to hold**.
  (a) Define $*$ on $\mathbf{Z}$ by $a * b = \max\{a, b\}$.
  (b) Define $*$ on $\mathbf{Z}$ by $a * b = a - b$.
  (c) Define $*$ on $\mathbf{Z}$ by $a * b = |ab|$.
  (d) Define $*$ on $\mathbf{R}^+$ by $a * b = ab$.

(3) Let $(G, \cdot)$ be a group. Define a new binary operation $*$ on G by the formula $a * b = b \cdot a$, for all $a, b \in G$.
  (a) Show that $(G, *)$ is a group.
  (b) Give examples to show that $(G, *)$ may or may not be the same as $(G, \cdot)$.

(4) Write out the multiplication table for $\mathbf{Z}_7^\times$.

(5) Let $G = \{x \in \mathbf{R} \mid x > 0 \text{ and } x \neq 1\}$. Define the operation $*$ on G by $a * b = a^{\ln b}$, for all $a, b \in G$. Prove that G is an abelian group under the operation $*$.

(6) Show that the set of all 2×2 matrices over $\mathbf{R}$ of the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$ with $m \neq 0$ forms a group under matrix multiplication. Furthermore, find all elements that commute with $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ in this group.

(7) Define $*$ on $\mathbf{R}$ by $a * b = a + b - 1$, for all $a, b \in \mathbf{R}$. Show that $(\mathbf{R}, *)$ is an abelian group.

(8) Let $S = \mathbf{R} - \{-1\}$. Define $*$ on S by $a * b = a + b + ab$, for all $a, b \in S$. Show that $(S, *)$ is an abelian group.

(9) Show that a nonabelian group must have at least five distinct elements.

(10) Let G be a group. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

(11) Let G be a group. Prove that if $x^2 = e$ for all $x \in G$, then G is abelian.

(12) Show that if G is a finite group with an even number of elements, then there must exist an element $a \in G$ with $a \neq e$ such that $a^2 = e$.