Homework 1

Due: May 15th (Friday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf, not as an email attachment (if needed, there are many online converters of jpg pictures to pdfs).
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.

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(1) Read §1.1 and §1.2 to make sure you understand the gcd, lcm, and Euclidean algorithm.

(2) Solve the following congruences.
  (a) $2x \equiv 1 \pmod{9}$
  (b) $10x \equiv 5 \pmod{15}$
  (c) $20x \equiv 12 \pmod{72}$

(3) Solve the following system of congruences.

$$x \equiv 15 \pmod{27} \qquad x \equiv 16 \pmod{20}$$

(4) (a) Make addition and multiplication tables for $\mathbf{Z}_4$.
  (b) Make multiplication table for $\mathbf{Z}_{12}^{\times}$.

(5) Find the multiplicative inverses of the given elements (if possible).
  (a) $[6]$ in $\mathbf{Z}_{15}$.
  (b) $[7]$ in $\mathbf{Z}_{15}$.

(6) Let $(a, n) = 1$. The smallest positive integer k such that $a^k \equiv 1 \pmod{n}$ is called the **multiplicative order** of $[a]$ in $\mathbf{Z}_n^{\times}$.
Find the multiplicative orders of $[5]$ and $[7]$ in $\mathbf{Z}_{16}^{\times}$ and show that their multiplicative orders both divide $\varphi(16)$.

(7) For $n = 12$ show that $\displaystyle\sum_{d|n} \varphi(d) = n$.

(8) Consider the following permutations in S_7.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \qquad \text{and} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

  (a) Write the following permutations as a product of disjoint cycles.

(i) $\sigma\tau$ (ii) $\tau\sigma$ (iii) σ^{-1} (iv) $\sigma\tau\sigma^{-1}$

  (b) Write σ and τ as products of transpositions.

(9) Write

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix}$$

as a product of disjoint cycles and as a product of transpositions. Find its inverse, and find its order.

(10) Find the order of each of the following permutations.
Hint: First write each permutation as a product of disjoint cycles.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 7 & 5 & 1 & 8 & 2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 9 & 8 & 7 & 3 & 4 & 6 & 1 & 2 \end{pmatrix}$

(11) Let $\sigma = (2396)(73259)(17)(487) \in S_9$.

(a) Is σ an even permutation or an odd permutation?

(b) What is the order of σ in S_9?