Final Exam

Exam Date: June 19th-20th (Friday-Saturday)

Exam Length: 150 minutes

- Please submit your work on Blackboard **before Saturday (6/20) 11:59 pm**.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- Open-book and Open-notes.
- **Honors Code:** No consulting any online sources. No consulting with each other.

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(0) Write the following honors code with your full name at the end.
I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code. As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam. **Full name**

(1) [*20 pts*] True or False: (*No need to show work to support your answer.*)

i) $\mathbf{R}$ is a group under multiplication. False: 0 has no inverse.

ii) A cyclic group is always abelian. True

iii) 8 is a unit in $\mathbf{Z}_{35}$. True

iv) If $|G| = 23$, then G must be isomorphic to $\mathbf{Z}_{23}$. True

v) The odd permutations in S_n form a normal subgroup of S_n. False: The odd permutations in S_n do not form a subgroup.

vi) $D_4 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. False: D_4 is nonabelian., while $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ is abelian.

vii) The order of gH in G/H is the smallest positive integer n such that $g^n = e$. False: $\ldots g^n \in H$.

viii) The product of an even number of disjoint cycles is an even permutation. False: $(12)(345)$ is odd.

ix) $\mathbf{Z}_{10} \times \mathbf{Z}_{10} \cong \mathbf{Z}_5 \times \mathbf{Z}_{20}$. False: There is no element of order 20 in $\mathbf{Z}_{10} \times \mathbf{Z}_{10}$, while $\mathbf{Z}_5 \times \mathbf{Z}_{20}$ does.

x) $\mathbf{Z}_{37}$ is a simple group. True

(2) (a) [*3 pts*] Solve the congruence $7x \equiv 1 \pmod{17}$. $x \equiv 5 \pmod{17}$

(b) [*3 pts*] Solve the congruence $12x \equiv 30 \pmod{54}$.

$\gcd(12, 54) = 6 | 30 \Rightarrow 2x \equiv 5 \pmod 9$. Since $2x \equiv 1 \pmod 9$ implies $x \equiv 5 \pmod 9$. Thus, $2x \equiv 5 \pmod 9 \Rightarrow x \equiv 25 \equiv 7 \pmod 9$. Therefore,

$$x \equiv 7, 16, 25, 34, 43, 52 \pmod{54}.$$

(c) [*4 pts*] Solve the system of congruences $\quad 5x \equiv 7 \pmod{12} \qquad x \equiv 13 \pmod{19}$.

$$\begin{bmatrix}1&0&12\\0&1&19\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&0&12\\-1&1&7\end{bmatrix} \rightsquigarrow \begin{bmatrix}2&-1&5\\-1&1&7\end{bmatrix} \rightsquigarrow \begin{bmatrix}2&-1&5\\-3&2&2\end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix}8&-5&1\\-3&2&2\end{bmatrix} \rightsquigarrow \begin{bmatrix}8&-5&1\\-19&12&0\end{bmatrix} \Rightarrow 8\cdot 12 + (-5)\cdot 19 = 1$$

Since $5x \equiv 7 \pmod{12} \Rightarrow x \equiv -1 \equiv 11 \pmod{12}$, by Chinese Remainder Theorem we have $x \equiv (11)(-95)+13(96) \pmod{12\cdot 19}$. That is, $x \equiv 203 \pmod{228}$.

(3) [*10 pts*] For the permutation $\sigma = \begin{pmatrix}1&2&3&4&5&6&7&8&9&10\\3&8&5&2&10&9&1&4&6&7\end{pmatrix}$:

(a) Write σ as a product of disjoint cycles.

$\sigma = (135\ 10\ 7)(284)(69)$

(b) Write σ as a product of transpositions.

$\sigma = (13)(35)(5\ 10)(10\ 7)(28)(84)(69) = (10\ 7)(57)(37)(17)(84)(24)(69)$

(c) Write σ^{-1} as a product of disjoint cycles.

$\sigma = (17\ 10\ 53)(248)(69)$ since disjoint cycles commute.

(d) Is σ even, odd, neither or both? Is σ^{-1} even, odd, neither or both?

odd

(e) What is the order of σ?

$\text{lcm}[5,3,2] = 30.$

(4) [*10 pts*] Let G be the set of nonzero rational numbers $\mathbf{Q}^\times$. Define a new multiplication by $a * b = \frac{ab}{5}$, for all $a, b \in G$. Show that $(G, *)$ is an abelian group.

(i) Closure: Trivial since $a, b \in \mathbf{Q}^\times$.

(ii) Associative: For any $a, b, c \in G$, we have

$$(a*b)*c = \frac{ab}{5} * c = \frac{abc}{25} = a * \frac{bc}{5} = a*(b*c)$$

commutative: $a * b = \frac{ab}{5} = \frac{ba}{5} = b * a$

(iii) Identity: The identity element is 5. For any $a \in \mathbf{Q}^\times$, we have $5 * a = \frac{5a}{5} = a$.

(iv) Inverses: For any $a \in \mathbf{Q}^\times$, its inverse is $\frac{25}{a} \in \mathbf{Q}^\times$ since $a \in \mathbf{Q}^\times$. In fact,

$$a * \frac{25}{a} = \frac{a \cdot 25/a}{5} = 5.$$

For parts (iii)-(iv), we only check one equation because of the commutativity.

(5) (a) [*3 pts*] Let $G = \langle a \rangle$ be a group of order 50. What is the order of $\langle a^{35} \rangle$?

The order of $\langle a^{35} \rangle$ is $\frac{50}{\gcd(35, 50)} = 10.$

(b) [*3 pts*] What is the order of $([18]_{20}, [25]_{30})$ in $\mathbf{Z}_{20} \times \mathbf{Z}_{30}$?

$o([18]_{20}) = \dfrac{20}{\gcd(18,20)} = 10$ and $o([25]_{30}) = \dfrac{30}{\gcd(25,30)} = 6$. Thus, we have

$$o(([18]_{20}, [25]_{30})) = \text{lcm}[10, 6] = 30.$$

(c) [*4 pts*] Let $G = \mathbf{Z}_{48}$. List all possible choice of $[k]_{48}$ such that $\langle [k]_{48} \rangle = \langle [20]_{48} \rangle$.

$\langle [k]_{48} \rangle = \langle [20]_{48} \rangle = \langle [4]_{48} \rangle \Rightarrow \gcd(k, 48) = 4 \Rightarrow \gcd(\frac{k}{4}, 12) = 1$. And so all possible choice of $[k]_{48}$ such that $\langle [k]_{48} \rangle = \langle [20]_{48} \rangle$ are

$$[4]_{48}, [20]_{48}, [28]_{48}, [44]_{48}.$$

(6) [*5 pts*] Let G be a non-cyclic group of order 27. Prove that $a^9 = e$ for all $a \in G$.

Proof. Since G is not cyclic, it follows from Lagrange's theorem that an element $a \in G$ can have order $1, 3$ or 9. Hence proved. □

(7) Let H be a subgroup of G. Let $N(H) = \{g \in G \mid gHg^{-1} = H\}$. Prove

(a) [*4 pts*] $N(H)$ is a subgroup of G.

Proof. $N(H)$ is nonempty since $eHe^{-1} = H$, i.e., $e \in N(H)$.
For any $a, b \in N(H)$, we have

$$abH(ab)^{-1} = a(bHb^{-1})a^{-1} = aHa^{-1} = H.$$

This implies that $ab \in H$. Finally, for any $a \in N(H)$ we have

$$H = (a^{-1}a)H(a^{-1}a) = a^{-1}(aHa^{-1})a = a^{-1}H(a^{-1})^{-1} \text{ since } aHa^{-1} = H.$$

This implies that $a^{-1} \in N(H)$. □

(b) [*4 pts*] H is a subgroup of $N(H)$.

Proof. It suffices to show that $H \subseteq N(H)$ since both H and $N(H)$ are subgroups of G. Thus, for any $h \in H$, we need to show $hHh^{-1} = H$.

$hHh^{-1} \subseteq H$: For any $h' \in H$, we have $hh'h^{-1} \in H$ since H is a subgroup.

$H \subseteq hHh^{-1}$: For any $h' \in H$, we have $h' = h(h^{-1}h'h)h^{-1} \in hHh^{-1}$ since $h^{-1}h'h \in H$.

□

(c) [*2 pts*] H is normal in $N(H)$.

Proof. For any $h \in H$ and $g \in N(H)$, we have $ghg^{-1} \in gHg^{-1} = H$. □

(8) [*5 pts*] Let N_1 and N_2 be normal subgroups of the group G and let $N_1 \cap N_2 = \{e\}$. Prove that $n_1 n_2 = n_2 n_1$ for all $n_1 \in N_1$ and $n_2 \in N_2$.

Proof. $n_2^{-1} n_1 n_2 n_1^{-1} = n_2^{-1}(n_1 n_2 n_1^{-1}) \in N_2$ since N_2 is a normal subgroup.

$n_2^{-1} n_1 n_2 n_1^{-1} = (n_2^{-1} n_1 n_2) n_1^{-1} \in N_1$ since N_1 is a normal subgroup.

This implies that $n_2^{-1} n_1 n_2 n_1^{-1} \in N_1 \cap N_2 = \{e\}$. That is, $n_2^{-1} n_1 n_2 n_1^{-1} = e$. So

$$n_2^{-1} n_1 n_2 n_1^{-1} = e \Rightarrow n_1 n_2 = n_2 n_1.$$

□

(9) Let G and H be groups. Define the function $\phi : G \times H \to G$ by

$$\phi((a, b)) = a, \text{ for all } (a, b) \in G \times H.$$

(a) [*3 pts*] Prove that ϕ is a group homomorphism and onto.

Proof. It is clear that ϕ is well-defined and onto. For any $(a_1, b_1), (a_2, b_2)$, we have $\phi((a_1, b_1), (a_2, b_2)) = \phi(a_1a_2, b_1b_2) = a_1a_2 = \phi((a_1, b_1))\phi((a_2, b_2))$. □

(b) [*2 pts*] Find $\ker(\phi)$.

$\ker(\phi) = \{(a, b) \in G \times H \mid \phi((a, b)) = a = e_G\} = e_G \times H$.

(10) (a) [*2 pts*] Let G be an abelian group. Let H be a subgroup of G. Prove that $aH = Ha$ for any $a \in G$.

Proof. Any subgroup of an abelian group is normal by commutativity. □

(b) [*4 pts*] List the cosets of $\langle [11]_{24} \rangle$ in $\mathbf{Z}_{24}^{\times}$.

$\mathbf{Z}_{24}^{\times} = \{[1]_{24}, [5]_{24}, [7]_{24}, [11]_{24}, [13]_{24}, [17]_{24}, [19]_{24}, [23]_{24}\}$

$\langle [11]_{24} \rangle = \{[1]_{24}, [11]_{24}\}$.

$[5]_{24}\langle [11]_{24} \rangle = \{[5]_{24}, [7]_{24}\}$.

$[13]_{24}\langle [11]_{24} \rangle = \{[13]_{24}, [23]_{24}\}$.

$[17]_{24}\langle [11]_{24} \rangle = \{[17]_{24}, [19]_{24}\}$.

(c) [*4 pts*] Prove that the factor group $\mathbf{Z}_{24}^{\times}/\langle [11]_{24} \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

Proof. From part (b), we know that the factor group $\mathbf{Z}_{24}^{\times}/\langle [11]_{24} \rangle$ has order 4. So it must be isomorphic to $\mathbf{Z}_4$ or $\mathbf{Z}_2 \times \mathbf{Z}_2$. Moreover, every non-identity element in the factor group has order 2. In particular, $[5]_{24}^2 = [1]_{24}, [13]_{24}^2 = [1]_{24}$, and $[17]_{24}^2 = [1]_{24}$. This implies that $\mathbf{Z}_{24}^{\times}/\langle [11]_{24} \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. □

(11) [*5 pts*] May you have a good summer! Stay safe!