Final Exam

Exam Date: June 19th-20th (Friday-Saturday)

Exam Length: 150 minutes

- Please submit your work on Blackboard **before Saturday (6/20) 11:59 pm**.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- Open-book and Open-notes.
- **Honors Code:** No consulting any online sources. No consulting with each other.

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(0) Write the following honors code with your full name at the end.
I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code. As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam. **Full name**

(1) [*20 pts*] True or False: (*No need to show work to support your answer.*)

i) $\mathbf{R}$ is a group under multiplication.

ii) A cyclic group is always abelian.

iii) 8 is a unit in $\mathbf{Z}_{35}$.

iv) If $|G| = 23$, then G must be isomorphic to $\mathbf{Z}_{23}$.

v) The odd permutations in S_n form a normal subgroup of S_n.

vi) $D_4 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

vii) The order of gH in G/H is the smallest positive integer n such that $g^n = e$.

viii) The product of an even number of disjoint cycles is an even permutation.

ix) $\mathbf{Z}_{10} \times \mathbf{Z}_{10} \cong \mathbf{Z}_5 \times \mathbf{Z}_{20}$.

x) $\mathbf{Z}_{37}$ is a simple group.

(2) (a) [*3 pts*] Solve the congruence $7x \equiv 1 \pmod{17}$.

(b) [*3 pts*] Solve the congruence $12x \equiv 30 \pmod{54}$.

(c) [*4 pts*] Solve the system of congruences $5x \equiv 7 \pmod{12}$ $\quad x \equiv 13 \pmod{19}$.

(3) [*10 pts*] For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 8 & 5 & 2 & 10 & 9 & 1 & 4 & 6 & 7 \end{pmatrix}$:

(a) Write σ as a product of disjoint cycles.

(b) Write σ as a product of transpositions.

(c) Write σ^{-1} as a product of disjoint cycles.

(d) Is σ even, odd, neither or both? Is σ^{-1} even, odd, neither or both?

(e) What is the order of σ?

(4) [*10 pts*] Let G be the set of nonzero rational numbers $\mathbf{Q}^\times$. Define a new multiplication by $a * b = \dfrac{ab}{5}$, for all $a, b \in G$. Show that $(G, *)$ is an abelian group.

(5) (a) [*3 pts*] Let $G = \langle a \rangle$ be a group of order 50. What is the order of $\langle a^{35} \rangle$?

(b) [*3 pts*] What is the order of $([18]_{20}, [25]_{30})$ in $\mathbf{Z}_{20} \times \mathbf{Z}_{30}$?

(c) [*4 pts*] Let $G = \mathbf{Z}_{48}$. List all possible choice of $[k]_{48}$ such that $\langle [k]_{48} \rangle = \langle [20]_{48} \rangle$.

(6) [*5 pts*] Let G be a non-cyclic group of order 27. Prove that $a^9 = e$ for all $a \in G$.

(7) Let H be a subgroup of G. Let $N(H) = \{g \in G \mid gHg^{-1} = H\}$. Prove

(a) [*4 pts*] $N(H)$ is a subgroup of G.

(b) [*4 pts*] H is a subgroup of $N(H)$.

(c) [*2 pts*] H is normal in $N(H)$.

(8) [*5 pts*] Let N_1 and N_2 be normal subgroups of the group G and let $N_1 \cap N_2 = \{e\}$. Prove that $n_1 n_2 = n_2 n_1$ for all $n_1 \in N_1$ and $n_2 \in N_2$.

(9) Let G and H be groups. Define the function $\phi : G \times H \to G$ by

$$\phi((a, b)) = a, \text{ for all } (a, b) \in G \times H.$$

(a) [*3 pts*] Prove that ϕ is a group homomorphism and onto.

(b) [*2 pts*] Find $\ker(\phi)$.

(10) (a) [*2 pts*] Let G be an abelian group. Let H be a subgroup of G. Prove that $aH = Ha$ for any $a \in G$.

(b) [*4 pts*] List the cosets of $\langle [11]_{24} \rangle$ in $\mathbf{Z}_{24}^\times$.

(c) [*4 pts*] Prove that the factor group $\mathbf{Z}_{24}^\times / \langle [11]_{24} \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

(11) [*5 pts*] May you have a good summer! Stay safe!