Exam I

Exam Date: May 26th (Tuesday)
Exam Length: 100 minutes

- Please submit your work on Blackboard between 9 am and 9 pm.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.

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(0) Write the following honors code with your full name at the end.
I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code. As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam. **Full name**

(1) [*10 pts*] Solve the following (system of) congruences.

(a) $5x \equiv 1 \pmod{13}$

(b) $12x \equiv 40 \pmod{88}$

(c) $x \equiv 14 \pmod{28}$ $\quad$ $x \equiv 15 \pmod{55}$

(2) [*8 pts*] Let $S = \{x \in \mathbf{R} \mid x \neq 3\}$. Define $*$ on S by

$$a * b = 12 - 3a - 3b + ab.$$

Prove that $(S, *)$ is a group.

(3) [*6 pts*] Let $(G, \cdot)$ be an abelian group with identity element e. Let

$$H = \{a \in G \mid a \cdot a \cdot a \cdot a = e\}.$$

Prove that H is a subgroup of G.

(4) (a) [*4 pts*] Find the cyclic subgroup of S_8 generated by the element $(135)(68)$.

(b) [*4 pts*] Find a subgroup H of S_8 that contains 15 elements.
You do not have to list all of the elements in H. Just prove it. That is,
Prove that H (the one you find) is a subgroup of order 15 in S_8.

(5) [*8 pts*] Let G be a group and the center of G is defined as

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}.$$

In Homework 3, we have showed that the center $Z(G)$ is a subgroup of G.
Let H be a subgroup of G. Prove that the set

$$HZ(G) = \{hz \mid h \in H, z \in Z(G)\}$$

is a subgroup of G.

(6) (a) [*3 pts*] What is the order of $([15]_{20}, [20]_{24})$ in $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$?

(b) [*3 pts*] What is the largest order of an element in $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$?
And use your answer to show that $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$ is not cyclic.

(c) [*4 pts*] Let $G = \mathbf{Z}_{10}^{\times} \times \mathbf{Z}_{10}^{\times}$. Let $H = \langle (3,7) \rangle$ and $K = \langle (7,7) \rangle$. Find HK in G.
Here, $(3,7)$ *means* $([3]_{10}, [7]_{10})$. *Just use the simplified notations in your answer.*