Homework 7

Due: Mar 23rd (Wednesday Class)

- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.

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(1) Find the orders of each of these permutations.

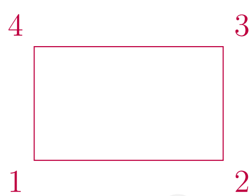
(a) (123)(2435)(132)

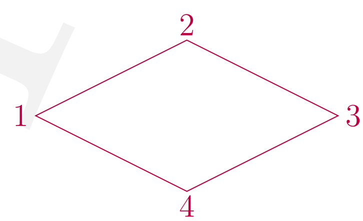
$= (1534)$, so order is 4.

(b) (136)(278)(42537)

$= (138256)(47)$, so order is $\text{lcm}[6, 2] = 6$.

(2) Find the permutations that correspond to the rigid motions of a rectangle that is not a square. Do the same for the rigid motions of a rhombus (diamond) that is not a square.





Rigid motions of the rectangle:
$(1), (14)(23), (12)(34), (13)(24)$.

Rigid motions of the rhombus:
$(1), (13), (24), (13)(24)$.

(3) Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that $ba^m = a^{-m}b$, for all $m \in \mathbf{Z}$.

For any positive integer m, we have

$$ba^m = (ba)a^{m-1} = (a^{-1}b)a^{m-1} = a^{-1}(ba)a^{m-2} = a^{-2}(ba^{m-2}) = \cdots = a^{-m}b.$$

If m is a negative integer, then $m = -|m|$, and so we have

$$ba^m = ba^{-|m|} = b(a^{-1})^{|m|} = (a^{-1})^{-|m|}b = (a^{-1})^m b = a^{-m}b.$$

It is trivial for $m = 0$. In conclusion, we have $ba^m = a^{-m}b$, for all $m \in \mathbf{Z}$.

(4) Find the order of each element of D_6.

We know that

$D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$, where $a^6 = e, b^2 = e, ba = a^{-1}b$.

And we also know that $o(a^j) = \dfrac{n}{\gcd(j, n)}$. Thus,

	e	a	a^2	a^3	a^4	a^5
order	1	6	3	2	3	6

Claim: All the remaining elements of the form $a^j b$ have the order 2 for $0 \le j < 6$.

$$(a^j b)^2 = a^j(ba^j)b \stackrel{!}{=} a^j(a^{-j}b)b = (a^j a^{-j})(bb) = ee = e.$$

$\stackrel{!}{=}$ holds because of Question (3). Note that the above claim holds for any D_n.

(5) Let $\tau = (abc)$ and let σ be any permutation. Show that $\sigma\tau\sigma^{-1} = (\sigma(a)\sigma(b)\sigma(c))$.

$\sigma\tau\sigma^{-1}(\sigma(a)) = \sigma(\tau(\sigma^{-1}\sigma(a))) = \sigma(\tau(a)) = \sigma(b)$: This implies $\sigma(a) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(b)$. Similarly, we can check that $\sigma(b) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(c)$ and $\sigma(c) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(a)$.

There is no need to consider other elements since they are fixed by τ. For example, take any $d \notin \{a, b, c\}$, then $\sigma\tau\sigma^{-1}(\sigma(d)) = \sigma\tau(d) = \sigma(d)$. □

(6) If $(12\cdots k)$ is a cycle of length k and σ is any permutation, then show that $\sigma(12\cdots k)\sigma^{-1} = (\sigma(1)\sigma(2)\cdots\sigma(k))$. (Hint: It is a generalization of Question (5).)

For any $1 \leq i < k$, we have

$$\sigma(12\cdots k)\sigma^{-1}(\sigma(i)) = \sigma((12\cdots k)(\sigma^{-1}\sigma(i))) = \sigma((12\cdots k)(i)) = \sigma(i+1).$$

And for $i = k$, we just take $i+1$ as 1. Again, with the same reason, there is no need to consider the other elements. □

(7) (a) In S_4, find the subgroup H generated by (123) and (23).

Since $(123) \in H$, $\langle(123)\rangle = \{(1), (123), (132)\} \subseteq H$.
Since $(23) \in H$, $\langle(23)\rangle = \{(1), (23)\} \subseteq H$.
Since H is a group, so by the closure axiom H contains

$$\{(1), (123), (132), (23), (123)(23) = (12), (132)(23) = (13)\}.$$

And this set is closed under multiplication. In fact, $H \cong S_3$. Thus,

$$H = \{(1), (123), (132), (23), (12), (13)\}.$$

(b) For $\sigma = (234)$, find the subgroup $\sigma H\sigma^{-1}$. (Hint: Use Question (5) or (6))

We need to compute $\sigma\tau\sigma^{-1}$ for each $\tau \in H$. First, $\sigma(1)\sigma^{-1} = \sigma\sigma^{-1} = (1)$. The calculations will be easier if we apply Question (5) or (6). In particular,

$$\begin{aligned}
\sigma(123)\sigma^{-1} &= (\sigma(1)\sigma(2)\sigma(3)) = (134)\\
\sigma(132)\sigma^{-1} &= (\sigma(1)\sigma(3)\sigma(2)) = (143)\\
\sigma(23)\sigma^{-1} &= (\sigma(2)\sigma(3)) = (34)\\
\sigma(12)\sigma^{-1} &= (\sigma(1)\sigma(2)) = (13)\\
\sigma(13)\sigma^{-1} &= (\sigma(1)\sigma(3)) = (14)
\end{aligned}$$

Thus, $\sigma H\sigma^{-1} = \{(1), (134), (143), (34), (13), (14)\}$.

(8)* Show that S_n is isomorphic to a subgroup of A_{n+2}. (Hint: Any odd permutation in S_n composite $\tau = (n+1\ n+2)$ is again an even permutations in S_{n+2}.)

Question (8) is a bonus question. It is optional for the students who are in Math 546. However, it is required for the students who are in Math 701I.*

Let $\tau = (n+1\ n+2)$. Consider the map $\phi\colon S_n \to A_{n+2}$ defined as follows:

$$\phi(\sigma) = \begin{cases} \sigma & \text{if } \sigma \text{ is even,} \\ \sigma\tau & \text{if } \sigma \text{ is odd.} \end{cases}$$

First of all, this map is well-defined since $\phi(\sigma) \in A_{n+2}$ for all $\sigma \in S_n$. Note that in this first case, i.e., when $\sigma \in S_n$ is even, you should think about $\phi(\sigma) = \sigma$ fixing $n+1$ and $n+2$ in A_{n+2}. Then consider the following map

$$\phi\colon S_n \to H, \text{ where } H = \phi(S_n).$$

Obviously, this map is onto. Then it suffice to show that

(a) $H = \phi(S_n)$ is a subgroup of A_{n+2}. Note that A_{n+2} is a finite group. Nonempty: $(1) = \phi((1)) \in H$; Closure: For any $\phi(\sigma_1), \phi(\sigma_2) \in H$, to show $\phi(\sigma_1)\phi(\sigma_2) \in H$.

  (i) If both of σ_1, σ_2 are even, then $\phi(\sigma_1)\phi(\sigma_2) = \sigma_1\sigma_2 = \phi(\sigma_1\sigma_2) \in H$.

  (ii) If both of σ_1, σ_2 are odd and so $\sigma_1\sigma_2$ is even, then $\phi(\sigma_1)\phi(\sigma_2) = \sigma_1\tau\sigma_2\tau \overset{!}{=} \sigma_1\sigma_2 = \phi(\sigma_1\sigma_2) \in H$ since $\tau\sigma_2 = \sigma_2\tau$ (they are disjoint) and so $\sigma_1\tau\sigma_2\tau = \sigma_1\sigma_2\tau^2 = \sigma_1\sigma_2$.

  (iii) If one of σ_1, σ_2 is odd and another one is even, say σ_1 is odd and σ_2 is even, then $\phi(\sigma_1)\phi(\sigma_2) = \sigma_1\tau\sigma_2 = \sigma_1\sigma_2\tau = \phi(\sigma_1\sigma_2) \in H$ since $\sigma_1\sigma_2$ is odd.

(b) ϕ preserves the product. It can be seen from the proof of above part (a).

(c) ϕ is one-to-one. If $\phi(\sigma) = (1)$, then we can see that σ must be (1). In fact,

  (i) If σ is even, then $\phi(\sigma) = \sigma$. Thus, $\sigma = (1)$ since $\phi(\sigma) = (1)$.

  (ii) If σ is odd, then $\phi(\sigma) = \sigma\tau$, which is impossible. Note that $\sigma\tau$ could never be (1) since σ fixing $n+1$ and $n+2$, but $\tau = (n+1\ n+2)$.

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