Homework 6

Due: Mar 16th (Wednesday Class)

- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.

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(1) Finish the proof of (⋆⋆) in Lecture Slides §3.5, #14/18. That is to say,

If $G_1 \cong H_1$ and $G_2 \cong H_2$, then $G_1 \times G_2 \cong H_1 \times H_2$.

Proof: Let $\theta_1 : G_1 \to H_1, \theta_2 : G_2 \to H_2$. Define $\phi : G_1 \times G_2 \to H_1 \times H_2$ by

$$\phi\big((x_1, x_2)\big) = \big(\theta_1(x_1), \theta_2(x_2)\big), \quad \text{for all } (x_1, x_2) \in G_1 \times G_2.$$

To show ϕ is a group isomorphism.

(2) Let G be a group and let $a \in G$ be an element of order 30. List the powers of a that have order 2, order 3 or order 5.

(3) Give the subgroup diagrams of the following groups.

(a) $\mathbf{Z}_{24}$

(b) $\mathbf{Z}_{36}$

(4) Which of $\mathbf{Z}_{18}^{\times}, \mathbf{Z}_{20}^{\times}$ are cyclic? (*Hint: Do not use The Primitive Root Theorem. Check Lecture Slides §3.5, #17/18*)

(5) Prove that $\mathbf{Z}_{10}^{\times}$ is not isomorphic to $\mathbf{Z}_{12}^{\times}$. (*Hint: Do not use The Primitive Root Theorem. Check Lecture Slides §3.5, #18/18*)

(6) You need to show work to support your conclusions. (*Hint: Check Lecture Slides §3.5, #14/18*)

(a) Is $\mathbf{Z}_3 \times \mathbf{Z}_{30}$ isomorphic to $\mathbf{Z}_6 \times \mathbf{Z}_{15}$?

(b) Is $\mathbf{Z}_9 \times \mathbf{Z}_{14}$ isomorphic to $\mathbf{Z}_6 \times \mathbf{Z}_{21}$?

(7) Let G be the set of all 3×3 matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. Show that if $a, b, c \in \mathbf{Z}_3$, then G is a group with exponent 3.

(8) Prove that any cyclic group with more than two elements has at least two different generators.

(9)* Let G be any group with no proper, nontrivial subgroups, and assume that G has more than one element. Prove that G must be isomorphic to $\mathbf{Z}_p$ for some prime p.

Question (9) is a bonus question. It is optional for the students who are in Math 546. However, it is required for the students who are in Math 701I.*