Homework 5

Due: Mar 2nd (Wednesday Class)

- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.

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(1) Show that the multiplicative group $\mathbf{Z}_7^\times$ is isomorphic to the additive group $\mathbf{Z}_6$.

(2) Show that the multiplicative group $\mathbf{Z}_8^\times$ is isomorphic to the group $\mathbf{Z}_2 \times \mathbf{Z}_2$.

(3) Show that $\mathbf{Z}_5^\times$ is not isomorphic to $\mathbf{Z}_8^\times$ by showing that the first group has an element of order 4 but the second group does not.

(4) Find two abelian groups of order 8 that are not isomorphic.

(5) Let G be any group, and let a be a fixed element of G. Define a function

$$\phi_a : G \to G \text{ by } \phi_a(x) = axa^{-1}, \text{ for all } x \in G.$$

Show that ϕ_a is an isomorphism.

(6) Let G be any group. Define $\phi : G \to G$ by $\phi(x) = x^{-1}$, for all $x \in G$.

(a) Prove that ϕ is one-to-one and onto.

(b) Prove that ϕ is an isomorphism if and only if G is abelian.

(7) Let $(G, \cdot)$ be a group. Define a new binary operation $*$ on G by the formula

$$a * b = b \cdot a, \text{ for all } a, b \in G.$$

Show that the group $(G, *)$ is isomorphic to the group $(G, \cdot)$.

(8)* Define $*$ on $\mathbf{R}$ by $a * b = a + b - 1$, for all $a, b \in \mathbf{R}$. Show that the group $(\mathbf{R}, *)$ is isomorphic to the group $(\mathbf{R}, +)$.

Question (8) is a bonus question. It is optional for the students who are in Math 546. However, it is required for the students who are in Math 701I.*