Homework 3

Due: Feb 9th (Wednesday class)

- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.

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(1) In the group $\mathrm{GL}_2(\mathbf{R})$ under matrix multiplication, find the order of each of the following elements.

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^6 = (-I_2)^2 = I_2$. Thus, the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ has order 6.

It easily follows from the direct computations to see that its order cannot be 4 or 5.

(b) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$ for all n. Thus $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ has infinite order.

(2) For each of the following groups, find all cyclic subgroups of the group.

(a) $(\mathbf{Z}_8, +_{[\ ]_8})$

$\mathbf{Z}_8 = \langle[1]\rangle = \langle[3]\rangle = \langle[5]\rangle = \langle[7]\rangle$ since $\mathbf{Z}_8^\times = \{[1],[3],[5],[7]\}$.
$\langle[2]\rangle = \langle[6]\rangle = \{[0],[2],[4],[6]\}$
$\langle[4]\rangle = \{[0],[4]\}$
$\langle[0]\rangle = \{[0]\}$

(b) $(\mathbf{Z}_{12}^\times, \cdot_{[\ ]_{12}})$

$\mathbf{Z}_{12}^\times = \{[1],[5],[7],[11]\} = \{[1],[5],[-5],[-1]\}$
$\langle[1]\rangle = \{[1]\}$
$\langle[5]\rangle = \{[1],[5]\}$
$\langle[7]\rangle = \{[1],[7]\}$
$\langle[11]\rangle = \{[1],[11]\}$
This implies that $\mathbf{Z}_{12}^\times$ is not a cyclic group.

(3) Find the cyclic subgroup of S_6 generated by the element $(123)(456)$.

$[(123)(456)]^2 = (123)^2(456)^2 = (132)(465)$ since (123) and (456) are disjoint.
$[(123)(456)]^3 = (123)^3(456)^3 = (1)$ since (123) and (456) are cycles of length 3
Thus, $\langle(123)(456)\rangle = \{(1),(123)(456),(132)(465)\}$.

(4) Let $G = \mathrm{GL}_3(\mathbf{R})$. Show that

$$H = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \right\}$$

is a subgroup of G.

(i) Closure: $\begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ b_1 & c_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ b_2 & c_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 + a_2 & 1 & 0 \\ b_1 + c_1 a_2 + b_2 & c_1 + c_2 & 1 \end{bmatrix} \in H.$

(ii) The identity matrix $I_3 \in H$ by letting $a = b = c = 0$.

(iii) Inverses: By part (i): $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b + ca & -c & 1 \end{bmatrix} \in H.$

(5) Prove that the intersection of any collection of subgroups of a group is again a subgroup.

Let G be a group and H_i be a subgroup of G for $i \in I$. (I is an index set) Then we need to show that $K = \cap_{i \in I} H_i$ is again a subgroup of G.

(a) Take any $a, b \in K \subseteq H_i$, for each i. Then $ab \in H_i$ since H_i is a subgroup. Thus, $ab \in K$ since i is arbitrary.
(b) The identity element $e \in H_i$ for each i, so $e \in K$.
(c) Take any $a \in K \subseteq H_i$, for each i. Then $a^{-1} \in H_i$ since H_i is a subgroup. Thus, $a^{-1} \in K$ since i is arbitrary.

(6) Prove that any cyclic group is abelian.

Let $\langle g \rangle$ be a cyclic group G. For any two elements $a, b \in G$, there exist $m, n \in \mathbf{Z}$ such that $a = g^m$ and $b = g^n$. Thus,

$$ab = g^m g^n = g^{m+n} = g^{n+m} = g^n g^m = ba.$$

(7) Let G be a non-cyclic group of order 8. Prove that $a^4 = e$ for all $a \in G$.

Proof. Since G is not cyclic, it follows from Lagrange's theorem that an element $a \in G$ can have order $1, 2$ or 4 since $8 = 2^3$. Hence proved. □

(8) Let G be a group. The set

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$$

of all elements that commute with every other element of G is called the **center** of G. Show that $Z(G)$ is a subgroup of G.

(a) If $x, y \in Z(G)$, then $xy \in Z(G)$ since by definition we have
$$(xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy) \text{ for all } g \in G.$$
(b) The identity element $e \in Z(G)$ since $eg = g = ge$ for all $g \in G$.
(c) If $x \in Z(G)$, then $x^{-1} \in Z(G)$. In fact, for all $g \in G$ we have
$$g = eg = (x^{-1}x)g = x^{-1}(xg) = x^{-1}(gx) = (x^{-1}g)x.$$
Thus, $gx^{-1} = x^{-1}g$ for all $g \in G$.

(9)* Show that if a group G has a unique element a of order 2, then $a \in Z(G)$. (Note that $Z(G)$, the center of G, is defined as in above Question (8).)

Question (9)* is a bonus question. It is optional for the students who are in Math 546. However, it is required for the students who are in Math 701I.

To show $a \in Z(G)$, it is equivalent to show that $ab = ba$ for all $b \in G$. Consider the element bab^{-1} for each $b \in G$, since $a^2 = e$ we have

$$(bab^{-1})^2 = (bab^{-1})(bab^{-1}) = bab^{-1}bab^{-1} = ba^2b^{-1} = beb^{-1} = e.$$

We omit the parentheses in the above calculations. There are two possibilities:

(a) If $bab^{-1} = e$, then $ba = b$. This implies $a = e$. We obtain a contradiction since $o(a) = 2$.

(b) If $bab^{-1} \neq e$, then $o(bab^{-1}) = 2$. So $bab^{-1} = a$ since the element a is the unique one in G with order 2. This implies $ba = ab$ for all $b \in G$. Thus, $a \in Z(G)$.