Homework 3

Due: Feb 9th (Wednesday class)

- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.

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(1) In the group $\mathrm{GL}_2(\mathbf{R})$ under matrix multiplication, find the order of each of the following elements.

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(2) For each of the following groups, find all cyclic subgroups of the group.

(a) $(\mathbf{Z}_8, +_{[\ ]_8})$

(b) $(\mathbf{Z}_{12}^{\times}, \cdot_{[\ ]_{12}})$

(3) Find the cyclic subgroup of S_6 generated by the element $(123)(456)$.

(4) Let $G = \mathrm{GL}_3(\mathbf{R})$. Show that

$$H = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \right\}$$

is a subgroup of G.

(5) Prove that the intersection of any collection of subgroups of a group is again a subgroup.

(6) Prove that any cyclic group is abelian.

(7) Let G be a non-cyclic group of order 8. Prove that $a^4 = e$ for all $a \in G$.

(8) Let G be a group. The set

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$$

of all elements that commute with every other element of G is called the **center** of G. Show that $Z(G)$ is a subgroup of G.

(9)* Show that if a group G has a unique element a of order 2, then $a \in Z(G)$. (Note that $Z(G)$, the center of G, is defined as in above Question (8).)

Question (9) is a bonus question. It is optional for the students who are in Math 546. However, it is required for the students who are in Math 701I.*