Math 546/701I—Final Exam

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(0) [**10 pts**] May you have a good summer holiday! Stay safe!

(1) [**20 pts**] True/False questions: Determine if each of the following is true or false. In each case, explain your answer in detail or give one counterexample if it is false.

i) $\mathbf{R}$ is a group under multiplication. False: 0 has no inverse.

ii) A cyclic group is always abelian. True

iii) $[9]_{35}$ is a unit in $\mathbf{Z}_{35}$. True

iv) If $|G| = 31$, then G must be isomorphic to $\mathbf{Z}_{31}$. True

v) A_n is a normal subgroup of S_n. True

vi) $D_4 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. False: D_4 is nonabelian., while $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ is abelian.

vii) The order of gH in G/H is the smallest positive integer n such that $g^n = e$. False: $\ldots g^n \in H$.

viii) The product of an even number of disjoint cycles is an even permutation. False: (12)(345) is odd.

ix) $\mathbf{Z}_{10} \times \mathbf{Z}_{10} \cong \mathbf{Z}_5 \times \mathbf{Z}_{20}$. False: There is no element of order 20 in $\mathbf{Z}_{10} \times \mathbf{Z}_{10}$, while $\mathbf{Z}_5 \times \mathbf{Z}_{20}$ does.

x) $\mathbf{Z}_{17}$ is a simple group. True

(2) [**15 pts**] Let G be the set of nonzero rational numbers $\mathbf{Q}^{\times}$. Define a new multiplication by $a * b = \dfrac{ab}{5}$, for all $a, b \in G$. Show that $(G, *)$ is an abelian group.

(i) Closure: Trivial since $a, b \in \mathbf{Q}^{\times}$.

(ii) Associative: For any $a, b, c \in G$, we have

$$(a * b) * c = \frac{ab}{5} * c = \frac{abc}{25} = a * \frac{bc}{5} = a * (b * c)$$

commutative: $a * b = \dfrac{ab}{5} = \dfrac{ba}{5} = b * a$

(iii) Identity: The identity element is 5. For any $a \in \mathbf{Q}^{\times}$, we have $5 * a = \dfrac{5a}{5} = a$.

(iv) Inverses: For any $a \in \mathbf{Q}^{\times}$, its inverse is $\dfrac{25}{a} \in \mathbf{Q}^{\times}$ since $a \in \mathbf{Q}^{\times}$. In fact,

$$a * \frac{25}{a} = \frac{a \cdot 25/a}{5} = 5.$$

For parts (iii)-(iv), we only check one equation because of the commutativity.

(3) (a) [**5 pts**] What is the order of $([18]_{20}, [25]_{30})$ in $\mathbf{Z}_{20} \times \mathbf{Z}_{30}$?

$o([18]_{20}) = \dfrac{20}{\gcd(18, 20)} = 10$ and $o([25]_{30}) = \dfrac{30}{\gcd(25, 30)} = 6$. Thus, we have

$$o(([18]_{20}, [25]_{30})) = \mathrm{lcm}[10, 6] = 30.$$

(b) [**5 pts**] Let $G = \mathbf{Z}_{48}$. List all possible choice of $[k]_{48}$ such that $\langle [k]_{48} \rangle = \langle [20]_{48} \rangle$.

$\langle [k]_{48} \rangle = \langle [20]_{48} \rangle = \langle [4]_{48} \rangle \Rightarrow \gcd(k, 48) = 4 \Rightarrow \gcd(\frac{k}{4}, 12) = 1$. And so all possible choice of $[k]_{48}$ such that $\langle [k]_{48} \rangle = \langle [20]_{48} \rangle$ are

$$[4]_{48}, [20]_{48}, [28]_{48}, [44]_{48}.$$

(4) Let H be a subgroup of G. Let $N(H) = \{g \in G \mid gHg^{-1} = H\}$. Prove

(a) [**8 pts**] $N(H)$ is a subgroup of G.

Proof. $N(H)$ is nonempty since $eHe^{-1} = H$, i.e., $e \in N(H)$.
For any $a, b \in N(H)$, we have

$$abH(ab)^{-1} = a(bHb^{-1})a^{-1} = aHa^{-1} = H.$$

This implies that $ab \in H$. Finally, for any $a \in N(H)$ we have

$$H = (a^{-1}a)H(a^{-1}a) = a^{-1}(aHa^{-1})a = a^{-1}H(a^{-1})^{-1} \text{ since } aHa^{-1} = H.$$

This implies that $a^{-1} \in N(H)$. □

(b) [**6 pts**] H is a subgroup of $N(H)$.

Proof. It suffices to show that $H \subseteq N(H)$ since both H and $N(H)$ are subgroups of G. Thus, for any $h \in H$, we need to show $hHh^{-1} = H$.

$hHh^{-1} \subseteq H$: For any $h' \in H$, we have $hh'h^{-1} \in H$ since H is a subgroup.

$H \subseteq hHh^{-1}$: For any $h' \in H$, we have $h' = h(h^{-1}h'h)h^{-1} \in hHh^{-1}$ since $h^{-1}h'h \in H$.

□

(c) [**6 pts**] H is normal in $N(H)$.

Proof. For any $h \in H$ and $g \in N(H)$, we have $ghg^{-1} \in gHg^{-1} = H$. □

(5) Let G and H be groups. Define the function $\phi : G \times H \to G$ by

$$\phi((a, b)) = a, \quad \text{for all } (a, b) \in G \times H.$$

(a) [**5 pts**] Prove that ϕ is a group homomorphism and onto.

Proof. It is clear that ϕ is well-defined and onto. For any $(a_1, b_1), (a_2, b_2)$, we have $\phi((a_1, b_1), (a_2, b_2)) = \phi(a_1 a_2, b_1 b_2) = a_1 a_2 = \phi((a_1, b_1))\phi((a_2, b_2))$. □

(b) [**5 pts**] Find $\ker(\phi)$.

$\ker(\phi) = \{(a, b) \in G \times H \mid \phi((a, b)) = a = e_G\} = \{e_G\} \times H$.

(6) (a) [**8 pts**] List the cosets of $\langle [11]_{24} \rangle$ in $\mathbf{Z}_{24}^{\times}$.

$\mathbf{Z}_{24}^{\times} = \{[1]_{24}, [5]_{24}, [7]_{24}, [11]_{24}, [13]_{24}, [17]_{24}, [19]_{24}, [23]_{24}\}$

$\langle [11]_{24} \rangle = \{[1]_{24}, [11]_{24}\}$.

$[5]_{24}\langle [11]_{24} \rangle = \{[5]_{24}, [7]_{24}\}$.

$[13]_{24}\langle [11]_{24} \rangle = \{[13]_{24}, [23]_{24}\}$.

$[17]_{24}\langle [11]_{24} \rangle = \{[17]_{24}, [19]_{24}\}$.

(b) [**7 pts**] Prove that the factor group $\mathbf{Z}_{24}^{\times}/\langle [11]_{24} \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.

Proof. From part (b), we know that the factor group $\mathbf{Z}_{24}^{\times}/\langle [11]_{24} \rangle$ has order 4. So it must be isomorphic to $\mathbf{Z}_4$ or $\mathbf{Z}_2 \times \mathbf{Z}_2$. Moreover, every non-identity element in the factor group has order 2. In particular, $[5]_{24}^2 = [1]_{24}, [13]_{24}^2 = [1]_{24}$, and $[17]_{24}^2 = [1]_{24}$. This implies that $\mathbf{Z}_{24}^{\times}/\langle [11]_{24} \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. □