Math 546/701I—Final Exam

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(0) [**10 pts**] May you have a good summer holiday! Stay safe!

(1) [**20 pts**] True/False questions: Determine if each of the following is true or false. In each case, explain your answer in detail or give one counterexample if it is false.

   i) $\mathbf{R}$ is a group under multiplication.

   ii) A cyclic group is always abelian.

   iii) $[9]_{35}$ is a unit in $\mathbf{Z}_{35}$.

   iv) If $|G| = 31$, then G must be isomorphic to $\mathbf{Z}_{31}$.

   v) A_n is a normal subgroup of S_n.

   vi) $D_4 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

   vii) The order of gH in G/H is the smallest positive integer n such that $g^n = e$.

   viii) The product of an even number of disjoint cycles is an even permutation.

   ix) $\mathbf{Z}_{10} \times \mathbf{Z}_{10} \cong \mathbf{Z}_5 \times \mathbf{Z}_{20}$.

   x) $\mathbf{Z}_{17}$ is a simple group.

(2) [**15 pts**] Let G be the set of nonzero rational numbers $\mathbf{Q}^\times$. Define a new multiplication by $a * b = \dfrac{ab}{5}$, for all $a, b \in G$. Show that $(G, *)$ is an abelian group.

(3) (a) [**5 pts**] What is the order of $([18]_{20}, [25]_{30})$ in $\mathbf{Z}_{20} \times \mathbf{Z}_{30}$?

(b) [**5 pts**] Let $G = \mathbf{Z}_{48}$. List all possible choice of $[k]_{48}$ such that $\langle [k]_{48} \rangle = \langle [20]_{48} \rangle$.

(4) Let H be a subgroup of G. Let $N(H) = \{g \in G \mid gHg^{-1} = H\}$. Prove

(a) [**8 pts**] $N(H)$ is a subgroup of G.

(b) [**6 pts**] H is a subgroup of $N(H)$.

(c) [**6 pts**] H is normal in $N(H)$.

(5) Let G and H be groups. Define the function $\phi : G \times H \to G$ by
$$\phi((a, b)) = a, \quad \text{for all } (a, b) \in G \times H.$$

(a) [**5 pts**] Prove that ϕ is a group homomorphism and onto.

(b) [**5 pts**] Find $\ker(\phi)$.

(6) (a) [**8 pts**] List the cosets of $\langle [11]_{24} \rangle$ in $\mathbf{Z}_{24}^{\times}$.

(b) [**7 pts**] Prove that the factor group $\mathbf{Z}_{24}^{\times}/\langle [11]_{24} \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.