Math 546/701I—Exam II

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(1) **[15 points]** True/False questions: Determine if each of the following is true or false. In each case, explain your answer in detail or give one counterexample if it is false.

(a) True or False: $4\mathbf{Z} \cong 8\mathbf{Z}$.

(b) True or False: Let σ be any permutation in S_n. Then σ^2 must be in A_n.

(c) True or False: Let p be a prime number. Then $\mathbf{Z}_p \times \mathbf{Z}_p \cong \mathbf{Z}_{p^2}$.

(d) True or False: Every subgroup of a non-cyclic group is non-cyclic.

(e) True or False: Two finite groups are isomorphic if they have the same order.

(2) **[12 points]** Let $G = \{x \in \mathbf{R} \mid x > 0 \text{ and } x \neq 1\}$, and define $*$ on G by

$$a * b = a^{\ln b} \qquad \text{for all } a, b \in G.$$

In Homework 2 (4), we have already shown that $(G, *)$ is an abelian group and the identity element is the natural number e.

Prove that $(G, *)$ is isomorphic to the group $\mathbf{R}^{\times}$ under the standard multiplication.

(3) (a) **[6 points]** Let G be a group and let $g \in G$ be an element of order 100. List all possible powers of g that have order 5. (Hint: Consider the cyclic subgroup $\langle g \rangle$ generated by g.)

(b) **[6 pts]** Let $G = \mathbf{Z}_{100}$. List all possible choice of $[k]_{100}$ such that $\langle [k]_{100} \rangle = \langle [15]_{100} \rangle$.

(c) **[6 points]** Give the subgroup diagram of $\mathbf{Z}_{100}$.

(4) **[15 points]** Recall that $D_n = \{a^k, a^k b \mid 0 \le k < n\}$, where $a^n = e, b^2 = e$, and $ba = a^{-1}b$. Moreover, in Homework 7 (3), we have already shown that $ba^m = a^{-m}b$ for all $m \in \mathbf{Z}$.

(a) **[3 points]** Show that $(a^k b)^2 = e$ for each $0 \le k < n$.

(b) **[8 points]** Find the order of each element of D_{10}. (Hint:Use part (a).)

(c) **[4 points]** Is D_{10} isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_5$? Show work to support your answer.