Math 546/701I—Exam I

Instructor: Shaoyun Yi Name: ____________________

(1) [**15 pts**] Let $S = \{x \in \mathbf{R} \mid x \neq 3\}$. Define $*$ on S by

$$a * b = 12 - 3a - 3b + ab.$$

Prove that $(S, *)$ is a group.

(i) Closure: We need to show $a * b \in S$ for any $a, b \in S$. That is, we need to show $a * b \neq 3$ for any real numbers $a \neq 3, b \neq 3$.
$a * b = 12 - 3a - 3b + ab = 3 + (3-a)(3-b) \neq 3$ since $(3-a)(3-b) \neq 0$. ✓

(ii) Associativity: For any $a, b, c \in S$, we need to show $(a * b) * c = a * (b * c)$.

$$\begin{aligned}(a * b) * c =& (12 - 3a - 3b + ab) * c \\ =& 12 - 3(12 - 3a - 3b + ab) - 3c + (12 - 3a - 3b + ab)c \\ =& -24 + 9a + 9b + 9c - 3ab - 3ac - 3bc + abc\end{aligned}$$

$$\begin{aligned}a * (b * c) =& a * (12 - 3b - 3c + bc) \\ =& 12 - 3a - 3(12 - 3b - 3c + bc) + a(12 - 3b - 3c + bc) \\ =& -24 + 9a + 9b + 9c - 3bc - 3ab - 3ac + abc\end{aligned}$$

(iii) Identity: The identity element $e = 4$.
$a * 4 = 12 - 3a - 12 + 4a = a$ and $4 * a = 12 - 12 - 3a + 4a = a$.

(iv) Inverses: The inverse of a is $\frac{8-3a}{3-a} \neq 3$. It is well defined since $a \neq 3$.

$$a * \frac{8-3a}{3-a} = 12 - 3a - 3\frac{8-3a}{3-a} + a\frac{8-3a}{3-a} = 12 - 3a + \frac{-24 + 9a + 8a - 3a^2}{3-a} = 4 ✓$$

$$\frac{8-3a}{3-a} * a = 12 - 3\frac{8-3a}{3-a} - 3a + \frac{8-3a}{3-a}a = 12 - 3a + \frac{-24 + 9a + 8a - 3a^2}{3-a} = 4 ✓$$

(2) (a) [**6 pts**] Find the cyclic subgroup of S_8 generated by the element $(135)(68)$.

Using the property that the disjoint cycles commute with each other makes your calculations simpler. Note that the order of $(135)(68)$ is $\mathrm{lcm}[3,2] = 6$.

$$((135)(68))^2 = (135)^2(68)^2 = (153)$$
$$((135)(68))^3 = (153)(135)(68) = (68)$$
$$((135)(68))^4 = (68)(135)(68) = (135)(68)^2 = (135)$$
$$((135)(68))^5 = (135)(135)(68) = (153)(68)$$
$$((135)(68))^6 = (153)(68)(135)(68) = (153)(135)(68)(68) = (1)$$

Thus, the cyclic subgroup of S_8 generated by the element $(135)(68)$ is

$$\langle (135)(68) \rangle = \{(1), (135), (153), (68), (135)(68), (153)(68)\}.$$

(b) [**6 pts**] Find a subgroup H of S_8 that contains 15 elements.

You do not have to list all of the elements in H. Just prove it. That is,

Prove that H (the one you find) is a subgroup of order 15 in S_8.

As we know that the order of a product of disjoint cycles is the least common multiple of their lengths, then the element $(12345)(678)$ is a desired example since $\mathrm{lcm}[3,5] = 15$. In particular, let $H = \langle (12345)(678) \rangle$. Since the cyclic subgroup H is generated by $(12345)(678)$, thus $|H| = |\langle (12345)(678) \rangle| = o((12345)(678)) = 15$.

(3) [**12 pts**] Let G be a group and the center of G is defined as

$$Z = \{x \in G \mid xg = gx \text{ for all } g \in G\}.$$

Note: We have already showed that Z is a subgroup of G *in Homework 3 (8).*
Let H be a subgroup of G. Prove that the set

$$HZ = \{hz \mid h \in H, z \in Z\}$$

is a subgroup of G.

(i) Closure: For $h_1z_1, h_2z_2 \in HZ$, we need to show that $(h_1z_1)(h_2z_2) \in HZ$.

$$(h_1z_1)(h_2z_2) = ((h_1z_1)h_2)z_2 = (h_1(z_1h_2))z_2 \overset{!}{=} (h_1(h_2z_1))z_2 = (h_1h_2)(z_1z_2)\checkmark$$

In the above calculation, $\overset{!}{=}$ holds by the definition of Z.
$(h_1z_1)(h_2z_2) = (h_1h_2)(z_1z_2) \in HZ$ since H and Z are subgroups of G.

(ii) Identity: The identity element $e \in HZ$ since $e = ee \in HZ$.

(iii) Inverses: For any element $hz \in HZ$, its inverse is $h^{-1}z^{-1} \in HZ$.

$$(hz)(h^{-1}z^{-1}) = hzh^{-1}z^{-1} = h(zh^{-1})z^{-1} \overset{!}{=} h(h^{-1}z)z^{-1} = (hh^{-1})(zz^{-1}) = e$$

$$(h^{-1}z^{-1})(hz) = h^{-1}z^{-1}hz = h^{-1}(z^{-1}h)z \overset{!}{=} h^{-1}(hz^{-1})z = (h^{-1}h)(z^{-1}z) = e$$

Or, in G, we have $(hz)^{-1} = z^{-1}h^{-1}$. So for $hz \in HZ$, we have $(hz)^{-1} = z^{-1}h^{-1} \overset{!}{=} h^{-1}z^{-1} \in HZ$. Here we see $z^{-1} \in Z$ since Z is a subgroup.

Another way: It is clear that $h^{-1}zh \overset{!}{=} h^{-1}hz = z \in Z$ for all $h \in H$ and $z \in Z$. Thus HZ is a subgroup of G.

(4) (a) [**5 pts**] What is the order of $([15]_{20}, [20]_{24})$ in $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$?

$o([15]_{20}) = o([-5]_{20}) = 4$ and $o([20]_{24}) = o([-4]_{24}) = 6$.
Thus, the order of $([15]_{20}, [20]_{24})$ is $\text{lcm}[4, 6] = 12$.

(b) [**8 pts**] What is the largest order of an element in $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$?
And then use your answer to show that $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$ is not cyclic.

In $\mathbf{Z}_{20}$, the possible orders are $1, 2, 4, 5, 10$, and 20.
In $\mathbf{Z}_{24}$, the possible orders are $1, 2, 3, 4, 6, 8, 12$, and 24.
The largest possible least common multiple we can have is $\text{lcm}[20, 24] = 120$.
So there is no element of order $|\mathbf{Z}_{20} \times \mathbf{Z}_{24}| = 480$ and the group is not cyclic.
Another way to see that $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$ is not cyclic since $\gcd(20, 24) \neq 1$.

(c) [**8 pts**] Let $G = \mathbf{Z}_{10}^{\times} \times \mathbf{Z}_{10}^{\times}$. Let $H = \langle(3, 7)\rangle$ and $K = \langle(7, 7)\rangle$. Find HK in G.
Here, $(3, 7)$ *means* $([3]_{10}, [7]_{10})$. *Use this simplified notations in your answer.*

$H = \langle(3, 7)\rangle = \{(3, 7)^m, m \in \mathbf{Z}\} = \{(1, 1), (3, 7), (9, 9), (7, 3)\}$ has order 4
$K = \langle(7, 7)\rangle = \{(7, 7)^m, m \in \mathbf{Z}\} = \{(1, 1), (7, 7), (9, 9), (3, 3)\}$ has order 4
$HK = \{(1, 1), (3, 7), (9, 9), (7, 3), (1, 9), (9, 1), (3, 3), (7, 7)\}$ has order $8 = 4 \cdot 4/2$