

# Homework 8

Due: June 16th (Wednesday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (**5 pts for each**) that will be graded. (1), (2), (3), (5), (7)

(1) Write down all homomorphisms from  $\mathbf{Z}_{24}$  to  $\mathbf{Z}_{18}$ .

Define  $\phi : \mathbf{Z}_{24} \rightarrow \mathbf{Z}_{18}$  by  $\phi([x]_{24}) = [mx]_{18}$  for some  $[m]_{18} \in \mathbf{Z}_{18}$ . In order for  $\phi$  to be well-defined, we need the condition that  $18|24m$ . That is,  $3|4m$ , and so  $3|m$  since  $\gcd(3, 4) = 1$ . Then, all the possible  $[m]_{18}$ 's are  $[0]_{18}, [3]_{18}, [6]_{18}, [9]_{18}, [12]_{18}$  and  $[15]_{18}$ . Thus, all homomorphisms from  $\mathbf{Z}_{24}$  to  $\mathbf{Z}_{18}$  are:

$$\begin{aligned}\phi_0([x]_{24}) &= [0]_{18} \\ \phi_3([x]_{24}) &= [3x]_{18} \\ \phi_6([x]_{24}) &= [6x]_{18} \\ \phi_9([x]_{24}) &= [9x]_{18} \\ \phi_{12}([x]_{24}) &= [12x]_{18} \\ \phi_{15}([x]_{24}) &= [15x]_{18}\end{aligned}$$

defined for all  $[x]_{24} \in \mathbf{Z}_{24}$ .

(2) Write down all homomorphisms from  $\mathbf{Z}$  to  $\mathbf{Z}_{12}$ , which are onto.

Every homomorphism  $\phi : \mathbf{Z} \rightarrow \mathbf{Z}_{12}$  is defined by  $\phi(x) = [mx]_{12}$  for  $[m]_{12} \in \mathbf{Z}_{12}$ . Moreover, the homomorphism  $\phi$  is onto. This implies that  $\phi$  sends the generator 1 in  $\mathbf{Z}$  to the generator  $[m]_{12}$  in  $\mathbf{Z}_{12}$ . As we know that  $[m]_{12}$  generates  $\mathbf{Z}_{12}$  if and only if  $[m]_{12} \in \mathbf{Z}_{12}^\times$ , i.e.,  $\gcd(m, 12) = 1$ . Thus,  $m = 1, 5, 7, 11$ . In conclusion, all homomorphisms from  $\mathbf{Z}$  onto  $\mathbf{Z}_{12}$  are:

$$\begin{aligned}\phi_1(x) &= [x]_{12} \\ \phi_5(x) &= [5x]_{12} \\ \phi_7(x) &= [7x]_{12} \\ \phi_{11}(x) &= [11x]_{12}\end{aligned}$$

defined for all  $x \in \mathbf{Z}$ .

(3) For the group homomorphism  $\phi : \mathbf{Z}_{15}^\times \rightarrow \mathbf{Z}_{15}^\times$  defined by  $\phi([x]) = [x]^2$  for all  $[x] \in \mathbf{Z}_{15}^\times$ , find the kernel and image of  $\phi$ .

$$\frac{[x]}{\phi([x]) = [x]^2} \mid \begin{array}{cccccccc} [1] & [2] & [4] & [7] & [8] & [11] & [13] & [14] \\ [1] & [4] & [1] & [4] & [4] & [1] & [4] & [1] \end{array}$$

Thus,  $\ker(\phi) = \{[1], [4], [11], [14]\}$  and  $\text{im}(\phi) = \{[1], [4]\}$ .

- (4) Which of the following functions are homomorphisms? You need to show work to support your answers.

(a)  $\phi : (\mathbf{R}^\times, \cdot) \rightarrow (\text{GL}_2(\mathbf{R}), \cdot_{\text{matrix}})$  defined by  $\phi(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ .

Yes:

(i) well-defined:  $\phi(a) \in \text{GL}_2(\mathbf{R})$  since  $a \in \mathbf{R}^\times$ .

(ii) For any  $a, b \in \mathbf{R}^\times$ , we have

$$\begin{aligned} \phi(a \cdot b) &= \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \\ &= \phi(a)\phi(b) \end{aligned}$$

(b)  $\phi : (\text{M}_2(\mathbf{R}), +_{\text{matrix}}) \rightarrow (\mathbf{R}, +)$  defined by  $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a$ .

Yes:

(i) well-defined: Trivial.

(ii) For any  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \text{M}_2(\mathbf{R})$ , we have

$$\begin{aligned} \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) &= \begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix} \\ &= a + a' \\ &= \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \phi\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) \end{aligned}$$

(c)  $\phi : (\text{GL}_2(\mathbf{R}), \cdot_{\text{matrix}}) \rightarrow (\mathbf{R}^\times, \cdot)$  defined by  $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$ .

No:  $\phi$  is *not* well-defined. e.g., let  $a = d = 1, b = c = 0$ , and so  $ab = 0 \notin \mathbf{R}^\times$ .

- (5) Let  $\phi : G_1 \rightarrow G_2$  and  $\theta : G_2 \rightarrow G_3$  be group homomorphisms. Prove that

(a)  $\theta\phi : G_1 \rightarrow G_3$  is a homomorphism.

(i) well-defined: For any  $a \in G_1$ ,  $\theta\phi(a) = \theta(\phi(a)) \in G_3$  since  $\phi(a) \in G_2$ .

(ii) For any  $a, b \in G_1$ , we have

$$\theta\phi(a * b) = \theta(\phi(a * b)) = \theta(\phi(a) \cdot \phi(b)) = \theta(\phi(a)) * \theta(\phi(b)) = \theta\phi(a) * \theta\phi(b).$$

(b)  $\ker(\phi) \subseteq \ker(\theta\phi)$ .

For any  $a \in \ker(\phi)$ , we have  $\theta\phi(a) = \theta(\phi(a)) = \theta(e_2) = e_3$ , and so  $a \in \ker(\theta\phi)$ . This proves  $\ker(\phi) \subseteq \ker(\theta\phi)$ .

- (6) Let  $G$  be a group, and let  $H$  be a normal subgroup of  $G$ . Show that for each  $g \in G$  and  $h \in H$  there exist  $h_1$  and  $h_2$  in  $H$  with  $gh = h_1g$  and  $hg = gh_2$ .

By definition of the normal subgroup, for each  $g \in G$  and  $h \in H$  we have  $ghg^{-1} \in H$ . Say,  $ghg^{-1} = h_1$ , and so  $gh = h_1g$ . Since  $G$  is a group,  $g^{-1} \in G$ . Then  $g^{-1}h(g^{-1})^{-1} = g^{-1}hg \in H$ . Say  $g^{-1}hg = h_2$ , and so  $hg = gh_2$ .

- (7) Recall that the center  $Z(G)$  of a group  $G$  is

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}.$$

Prove that the center of any group is a normal subgroup.

For any  $a \in Z(G)$  (it is already a subgroup of  $G$ ) and any  $g \in G$ , we have

$$gag^{-1} = gg^{-1}a = ea = a \in Z(G).$$

Thus, the center of any group is a normal subgroup.

- (8) Prove that the intersection of two normal subgroups is a normal subgroup.

Optional: This is a bonus question. (5 points)

Let  $H_1$  and  $H_2$  be two normal subgroups of  $G$ . Let  $H = H_1 \cap H_2$ . It is also easy to see that  $H$  is a subgroup of  $G$ . It suffices to show that  $H$  is normal. Let  $h$  be any element in  $H$  and  $g$  be any element in  $G$ . Then we have

$$\begin{aligned} ghg^{-1} &\in H_1 \text{ since } h \in H_1 \text{ and } H_1 \text{ is a normal subgroup of } G; \\ ghg^{-1} &\in H_2 \text{ since } h \in H_2 \text{ and } H_2 \text{ is a normal subgroup of } G. \end{aligned}$$

This implies that  $ghg^{-1} \in H_1 \cap H_2 = H$ . Thus,  $H = H_1 \cap H_2$  is again a normal subgroup of  $G$ .