Homework 7

Due: June 12th (Saturday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions **(5 pts for each)** that will be graded.
  (1), (3), (4), (5), (7)

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(1) Find the orders of each of these permutations.

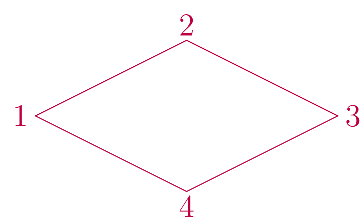
(a) (123)(2435)(132)

$= (1534)$, so order is 4.

(b) (136)(278)(42537)

$= (138256)(47)$, so order is $\text{lcm}[6, 2] = 6$.

(2) Find the permutations that correspond to the rigid motions of a rectangle that is not a square. Do the same for the rigid motions of a rhombus (diamond) that is not a square.



Rigid motions of the rectangle:
(1), (14)(23), (12)(34), (13)(24).

Rigid motions of the rhombus:
(1), (13), (24), (13)(24).

(3) Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that $ba^m = a^{-m}b$, for all $m \in \mathbf{Z}$.

For any positive integer m, we have

$$ba^m = (ba)a^{m-1} = (a^{-1}b)a^{m-1} = a^{-1}(ba)a^{m-2} = a^{-2}(ba^{m-2}) = \cdots = a^{-m}b.$$

If m is a negative integer, then $m = -|m|$, and so we have

$$ba^m = ba^{-|m|} = b(a^{-1})^{|m|} = (a^{-1})^{-|m|}b = (a^{-1})^m b = a^{-m}b.$$

It is trivial for $m = 0$. In conclusion, we have $ba^m = a^{-m}b$, for all $m \in \mathbf{Z}$.

(4) Find the order of each element of D_6.

We know that

$D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$, where $a^6 = e, b^2 = e, ba = a^{-1}b$.

And we also know that $o(a^j) = \dfrac{n}{\gcd(j, n)}$. Thus,

	e	a	a^2	a^3	a^4	a^5
order	1	6	3	2	3	6

Claim: All the remaining elements of the form $a^j b$ have the order 2 for $0 \le j < 6$.

$$(a^j b)^2 = a^j (ba^j) b \overset{!}{=} a^j (a^{-j} b) b = (a^j a^{-j})(bb) = ee = e.$$

$\overset{!}{=}$ holds because of Question (3). Note that the above claim holds for any D_n.

(5) Let $\tau = (abc)$ and let σ be any permutation. Show that $\sigma\tau\sigma^{-1} = (\sigma(a)\sigma(b)\sigma(c))$.

$\sigma\tau\sigma^{-1}(\sigma(a)) = \sigma(\tau(\sigma^{-1}\sigma(a))) = \sigma(\tau(a)) = \sigma(b)$: This implies $\sigma(a) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(b)$. Similarly, we can check that $\sigma(b) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(c)$ and $\sigma(c) \overset{\sigma\tau\sigma^{-1}}{\mapsto} \sigma(a)$.

There is no need to consider other elements since they are fixed by τ. For example, take any $d \notin \{a, b, c\}$, then

$$\sigma\tau\sigma^{-1}(\sigma(d)) = \sigma\tau(d) = \sigma(d).$$

□

(6) In general, if $(12\cdots k)$ is a cycle of length k and σ is any permutation, then show that

$$\sigma(12\cdots k)\sigma^{-1} = (\sigma(1)\sigma(2)\cdots\sigma(k)).$$

This is just a generalization of Question (5). For any $1 \le i < k$, we have

$$\sigma(12\cdots k)\sigma^{-1}(\sigma(i)) = \sigma((12\cdots k)(\sigma^{-1}\sigma(i))) = \sigma((12\cdots k)(i)) = \sigma(i+1).$$

And for $i = k$, we just take $i+1$ as 1. Again, with the same reason, there is no need to consider the other elements. □

(7) (a) In S_4, find the subgroup H generated by (123) and (23).

Since $(123) \in H$, $\langle(123)\rangle = \{(1), (123), (132)\} \subseteq H$.
Since $(23) \in H$, $\langle(23)\rangle = \{(1), (23)\} \subseteq H$.
Since H is a group, so by the closure axiom H contains

$$\{(1), (123), (132), (23), (123)(23) = (12), (132)(23) = (13)\}.$$

And this set is closed under multiplication. In fact, $H \cong S_3$. Thus,

$$H = \{(1), (123), (132), (23), (12), (13)\}.$$

(b) For $\sigma = (234)$, find the subgroup $\sigma H \sigma^{-1}$. (Hint: Use (5) or (6) above)

We need to compute $\sigma\tau\sigma^{-1}$ for each $\tau \in H$. First, $\sigma(1)\sigma^{-1} = \sigma\sigma^{-1} = (1)$. The calculations will be easier if we apply Question (5) or (6). In particular,

$$\begin{aligned}
\sigma(123)\sigma^{-1} &= (\sigma(1)\sigma(2)\sigma(3)) = (134)\\
\sigma(132)\sigma^{-1} &= (\sigma(1)\sigma(3)\sigma(2)) = (143)\\
\sigma(23)\sigma^{-1} &= (\sigma(2)\sigma(3)) = (34)\\
\sigma(12)\sigma^{-1} &= (\sigma(1)\sigma(2)) = (13)\\
\sigma(13)\sigma^{-1} &= (\sigma(1)\sigma(3)) = (14)
\end{aligned}$$

Thus, $\sigma H \sigma^{-1} = \{(1), (134), (143), (34), (13), (14)\}$.