Homework 6

Due: June 5th (Saturday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (**5 pts for each**) that will be graded.

---

(1) Finish the proof of $(\star\star)$ in Lecture Slides §3.5, #14/18.

(2) Let G be a group and let $a \in G$ be an element of order 30. List the powers of a that have order 2, order 3 or order 5.

(3) Give the subgroup diagrams of the following groups.
  (a) $\mathbf{Z}_{24}$
  (b) $\mathbf{Z}_{36}$

(4) Which of $\mathbf{Z}_{18}^{\times}, \mathbf{Z}_{20}^{\times}$ are cyclic? (*Do not use The Primitive Root Theorem.*)

(5) Prove that $\mathbf{Z}_{10}^{\times}$ is not isomorphic to $\mathbf{Z}_{12}^{\times}$. (*Do not use The Primitive Root Theorem.*)

(6) You need to show work to support your conclusions.
  (a) Is $\mathbf{Z}_3 \times \mathbf{Z}_{30}$ isomorphic to $\mathbf{Z}_6 \times \mathbf{Z}_{15}$?
  (b) Is $\mathbf{Z}_9 \times \mathbf{Z}_{14}$ isomorphic to $\mathbf{Z}_6 \times \mathbf{Z}_{21}$?

(7) Let G be the set of all 3×3 matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. Show that if $a, b, c \in \mathbf{Z}_3$, then G is a group with exponent 3.

(8) Prove that any cyclic group with more than two elements has at least two different generators.

(9) Let G be any group with no proper, nontrivial subgroups, and assume that G has more than one element. Prove that G must be isomorphic to $\mathbf{Z}_p$ for some prime p.

Optional: This is a bonus question. (5 points)