Homework 4

Due: May 29th (Saturday), 11:59 pm

- Please submit your work on Blackboard.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- No late work will be accepted.
- There are five randomly picked questions (2 pts for each) that will be graded.
  (1), (2), (4), (6), (7)

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(1) Find HK in $\mathbf{Z}_{16}^{\times}$, if $H = \langle[3]\rangle$ and $K = \langle[5]\rangle$.

$|\mathbf{Z}_{16}^{\times}| = \varphi(16) = 8$; $H = \langle[3]\rangle = \{[1], [3], [9], [11]\}$ and $K = \langle[5]\rangle = \{[1], [5], [9], [13]\}$

$HK = \mathbf{Z}_{16}^{\times} = \{[1], [3], [5], [7], [9], [11], [13], [15]\}$.

(2) Find the order of the element $([9]_{12}, [15]_{18})$ in the group $\mathbf{Z}_{12} \times \mathbf{Z}_{18}$.

$o([9]_{12}) = o([-3]_{12}) = o([3]_{12}) = 4$ in $\mathbf{Z}_{12}$ and $o([15]_{18}) = o([-3]_{18}) = o([3]_{18}) = 6$ in $\mathbf{Z}_{18}$. Here, we can also apply $\langle[a]_n\rangle = \langle[d]_n\rangle$, where $d = (a, n)$. Thus, $o(([9]_{12}, [15]_{18})) = \text{lcm}[4, 6] = 12$.

(3) Prove that if G_1 and G_2 are abelian groups, then the direct product $G_1 \times G_2$ is abelian.

(Assume that $(G_1, *)$ and $(G_2, \cdot)$ are abelian groups.)

For any two elements $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$, we have

$$(a_1, a_2)(b_1, b_2) = (a_1 * b_1, a_2 \cdot b_2) = (b_1 * a_1, b_2 \cdot a_2) = (b_1, b_2)(a_1, a_2).$$

(4) Construct an abelian group of order 12 that is not cyclic.

$\mathbf{Z}_2 \times \mathbf{Z}_6$ is abelian by Question (3). Since $(2, 6) = 2 \neq 1$, it is not cyclic. Here, we use the fact that $\mathbf{Z}_n \times \mathbf{Z}_m$ is cyclic if and only if $(n, m) = 1$.

(5) Construct a group of order 12 that is not abelian.

$\mathbf{Z}_2 \times S_3$ is not abelian since S_3 is not abelian. For example, $([0], (123))([0], (12)) = ([0], (13))$, but $([0], (12))([0], (123)) = ([0], (23))$.

(6) Let G_1 and G_2 be groups, with subgroups H_1 and H_2, respectively. Show that

$$\{(x_1, x_2) \mid x_1 \in H_1, x_2 \in H_2\}$$

is a subgroup of the direct product $G_1 \times G_2$.

Let $(G_1, *)$ and $(G_2, \cdot)$ be groups and let $S = \{(x_1, x_2) \mid x_1 \in H_1, x_2 \in H_2\}$.

(i) For $(x_1, x_2), (y_1, y_2) \in S$, we have $(x_1, x_2)(y_1, y_2) = (x_1 * x_2, y_1 \cdot y_2) \in S$ since H_1 and H_2 are the subgroups of G_1 and G_2, respectively.

(ii) The identity element $e = (e_1, e_2) \in S$, where e_i is the identity element of H_i (and also of G_i) for $i = 1, 2$.

(iii) Inverses: $(x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1}) \in S$. (Easy to check)

(7) Let G_1 and G_2 be groups, and let G be the direct product $G_1 \times G_2$. Let $H = \{(x_1, x_2) \in G_1 \times G_2 \mid x_2 = e_2\}$ and let $K = \{(x_1, x_2) \in G_1 \times G_2 \mid x_1 = e_1\}$.

(a) Show that H and K are subgroups of G.

We will show H is a subgroup of G and the proof for K should be similar.

(i) $(x_1, e_2), (y_1, e_2) \in H$, we have $(x_1, e_2)(y_1, e_2) = (x_1 * y_1, e_2 \cdot e_2) \in H$.

(ii) The identity element $(e_1, e_2) \in H$.

(iii) For $(x_1, e_2) \in H$, its inverse is $(x_1^{-1}, e_2) \in H$.

(b) Show that $HK = KH = G$.

$HK = KH$: For any element $(x_1, e_2) \in H$ and any element $(e_1, x_2) \in K$, we have

$$\begin{aligned}(x_1, e_2)(e_1, x_2) =& (x_1 * e_1, e_2 \cdot x_2) \\ =& (x_1, x_2) \in G \\ =& (e_1 * x_1, x_2 \cdot e_2) \\ =& (e_1, x_2)(x_1, e_2).\end{aligned}$$

$HK \subseteq G$: It is clear from above computation.

$G \subseteq HK$: For any element $(x_1, x_2) \in G$ for $x_1 \in G_1$ and $x_2 \in G_2$, we can write it as $(x_1, x_2) = (x_1 * e_1, e_2 \cdot x_2) = (x_1, e_2)(e_1, x_2)$, which is in HK.

(c) Show that $H \cap K = \{(e_1, e_2)\}$.

$\{(e_1, e_2)\} \subseteq H \cap K$: By definition, we have $(e_1, e_2) \in H$ and $(e_1, e_2) \in K$.

$H \cap K \subseteq \{(e_1, e_2)\}$: For any element $(x_1, x_2) \in H \cap K$, we have

$$\begin{cases}(x_1, x_2) \in H \Rightarrow x_2 = e_2 \\ (x_1, x_2) \in K \Rightarrow x_1 = e_1\end{cases} \implies (x_1, x_2) = (e_1, e_2).$$

(8) Let F be a field, and let H be the subset of $\mathrm{GL}_2(F)$ consisting of all invertible upper triangular matrices. Show that H is a subgroup of $\mathrm{GL}_2(F)$.

(i) Let $\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \in H$. In particular, $a_1 d_1 \neq 0, a_2 d_2 \neq 0$. Then

$$\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{bmatrix} \in H.$$

This is because the determinant of the product is $a_1 a_2 d_1 d_2 \neq 0$.

(ii) The identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$.

(iii) For any element $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in H$, its inverse is $\begin{bmatrix} 1/a & -b/(ad) \\ 0 & 1/d \end{bmatrix} \in H$.