Exam II

Exam Date: June 8th (Tuesday)
Exam Length: 100 minutes

- Please submit your work on Blackboard **between 09:00 am and 11:59 pm**.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- You will be allowed to use your notes and the book during the exam.
- **No collaborations are allowed.**
- **No consulting any online sources is allowed.**
- **No late work will be accepted.**
- Total score: *100 points.*

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(0) [5 points] Write the following honors code.
I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code. As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam. Your name

(1) [**15 points**] True or False:

(a) $3\mathbf{Z} \cong 9\mathbf{Z}$.

True. Both are infinite and cyclic.

(b) Let p be a prime number. Then $\mathbf{Z}_p \times \mathbf{Z}_p \cong \mathbf{Z}_{p^2}$.

False. $\mathbf{Z}_{p^2}$ is cyclic but $\mathbf{Z}_p \times \mathbf{Z}_p$ is not.

(c) Every subgroup of a non-cyclic group is non-cyclic.

False. For example, $S_3, \mathbf{Z}_2 \times \mathbf{Z}_2$.

(d) Two finite non-cyclic groups are isomorphic if they have the same order.

False. For example, $\mathbf{Z}_2 \times \mathbf{Z}_4, \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

(e) Let σ be any permutation in S_n. Then σ^2 must be in A_n.

True. σ^2 can be always written as a product of an even number of transpositions.

(2) [**20 points**] Let $G = \{x \in \mathbf{R} \mid x > 0 \text{ and } x \neq 1\}$, and define $*$ on G by $a * b = a^{\ln b}$ for all $a, b \in G$. In Homework 2 (4), we have already shown that $(G, *)$ is an abelian group and the identity element is the natural number e. Prove that the group $(G, *)$ is isomorphic to the group $\mathbf{R}^\times$ under the standard multiplication.

Define a function $\phi : \mathbf{R}^\times \to G$ by $\phi(y) = e^y$ for all $y \in \mathbf{R}^\times$. It is well-defined.

$$\phi(y) = e^y > 0 \text{ and } e^y \neq 1 \text{ since } y \in \mathbf{R}^\times. \text{ That is, } \phi(y) \in G \text{ for all } y \in \mathbf{R}^\times.$$

Moreover, we define $\phi^{-1} : G \to \mathbf{R}^\times$ by $\phi^{-1}(x) = \ln x$ for all $x \in G$. ✓
To show that ϕ is one-to-one and onto, we need to verify that ϕ^{-1} is the inverse function of ϕ. In fact, for all $x \in G$ and all $y \in \mathbf{R}^\times$, we have

$$\phi(\phi^{-1}(x)) = \phi(\ln x) = e^{\ln x} = x \quad \text{and} \quad \phi^{-1}(\phi(y)) = \phi^{-1}(e^y) = \ln(e^y) = y.$$

For any two elements $y_1, y_2 \in \mathbf{R}^\times$, we have

$$\phi(y_1 \cdot y_2) = e^{y_1 \cdot y_2} = (e^{y_1})^{y_2} = (e^{y_1})^{\ln(e^{y_2})} = e^{y_1} * e^{y_2} = \phi(y_1) * \phi(y_2).$$

This shows that ϕ respects the two operations. Thus, ϕ is an isomorphism.

(3) (a) **[8 points]** Let G be a group and let $g \in G$ be an element of order 100. List all possible powers of g that have order 5.

For any integer k, we have $\langle g^k \rangle = \langle g^d \rangle$ with $d = \gcd(k, 100)$. And $o(g^j) = |\langle g^k \rangle| = |\langle g^d \rangle| = \dfrac{100}{d} = \dfrac{100}{\gcd(k,100)} = 5$. So, $\gcd(k, 100) = 20$. It is equivalent to

$$\gcd\left(\frac{k}{20}, 5\right) = 1 \Rightarrow \frac{k}{20} = 1, 2, 3, 4 \Rightarrow k = 20, 40, 60, 80.$$

(b) **[8 points]** Let $G = \mathbf{Z}_{100}$. List all possible choice of $[k]_{100}$ such that $\langle [k]_{100} \rangle = \langle [15]_{100} \rangle$.

$\langle [k]_{100} \rangle = \langle [15]_{100} \rangle = \langle [5]_{100} \rangle$ since $\gcd(15, 100) = 5$. It follows that

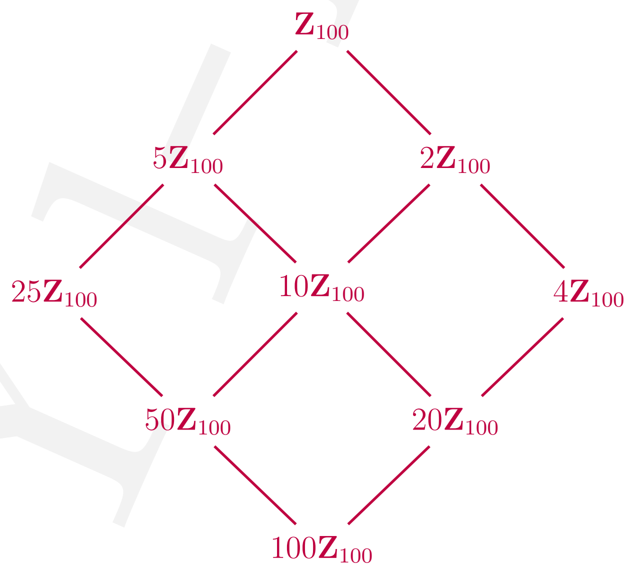
$$\langle [k]_{100} \rangle = \langle [5]_{100} \rangle \Leftrightarrow \gcd(k, 100) = 5 \Leftrightarrow \gcd\left(\frac{k}{5}, 20\right) = 1.$$

Thus, $\dfrac{k}{5} = 1, 3, 7, 9, 11, 13, 17, 19$. In conclusion, the possible choices are

$$k = 5, 15, 35, 45, 55, 65, 85, 95.$$

(c) **[8 points]** Give the subgroup diagram of $\mathbf{Z}_{100}$.

$100 = 2^2 5^2$: Any divisor $d = 2^i 5^j$, where $i = 0, 1, 2$ and $j = 0, 1, 2$.



(4) **[24 points]** Let $D_n = \{a^k, a^k b \mid 0 \le k < n\}$, where $a^n = e, b^2 = e$, and $ba = a^{-1}b$. Moreover, in Homework 7 (3), we have shown that $ba^m = a^{-m}b$ for all $m \in \mathbf{Z}$.

(a) **[6 points]** Show that $(a^k b)^2 = e$ for each $0 \le k < n$.

$(a^k b)^2 = (a^k b)(a^k b) = a^k (b a^k) b = a^k (a^{-k} b) b = (a^k a^{-k})(bb) = ee = e.$

(b) **[12 points]** Find the order of each element of D_{10}.

First, we know that $o(a^k) = \dfrac{10}{\gcd(k, 10)}$. Thus,

a^k	e	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9
order	1	10	5	10	5	2	5	10	5	10

It follows from Part (a) that all the remaining elements of the form $a^k b$ have the order 2 since $a^k b \neq e$. That is,

$a^k b$	b	ab	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b	a^8b	a^9b
order	2	2	2	2	2	2	2	2	2	2

(c) [**6 points**] Is D_{10} isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_5$? Show work to support your answer.

No. $\mathbf{Z}_4 \times \mathbf{Z}_5$ is cyclic but D_{10} is not. Or there is an element of order 4 in $\mathbf{Z}_4 \times \mathbf{Z}_5$ but D_{10} has none.

(5) [**12 points**] Let G be a finite group of order 125 with the identity element e. Assume that G contains an element a with $a^{25} \neq e$. Prove that G is cyclic.

Let $H = \langle a \rangle$. It is clear that H is a subgroup of G since $a \in G$. By Lagrange's Theorem, the possible orders of H are the divisors of $|G| = 125$. That is, $|H| = 1, 5, 25$, or 125.

Claim: $|H| = 125$.

If $|H| = 1, 5$, or 25, then $a^{25} = e$. This is a contradiction since $a^{25} \neq e$. $\square_{\text{Claim}}$

That is, $H = \langle a \rangle = G$. Therefore, G is cyclic. $\square$