Exam I

Exam Date: May 24th (Monday)
Exam Length: 100 minutes

- Please submit your work on Blackboard **between 09:00 am and 11:59 pm**.
- You are required to submit your work as a single pdf.
- Please make sure your handwriting is clear enough to read. Thanks.
- You will be allowed to use your notes and the book during the exam.
- **No collaborations are allowed.**
- **No consulting any online sources is allowed.**
- **No late work will be accepted.**
- Total score: *100 points*.

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(1) Solve the following congruences.

  (a) [**10 points**] $5x \equiv 1 \pmod{13}$

  (b) [**10 points**] $12x \equiv 40 \pmod{88}$

(2) [**20 points**] Let $S = \{x \in \mathbf{R} \mid x \neq 3\}$. Define $*$ on S by

$$a * b = 12 - 3a - 3b + ab.$$

Prove that $(S, *)$ is a group.

(3) [**15 points**] Let $(G, \cdot)$ be an abelian group with identity element e. Let

$$H = \{a \in G \mid a \cdot a \cdot a \cdot a = e\}.$$

Prove that H is a subgroup of G.

(4) (a) [**6 points**] Find the cyclic subgroup of S_8 generated by the element $(135)(68)$.

  (b) [**7 points**] Find a subgroup H of S_8 that contains 15 elements. You do not have to list all of the elements in H. Just prove it. That is, Prove that H (the one you find) is a subgroup of order 15 in S_8.

(5) [**15 points**] Let G be a group and the center of G is defined as

$$Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}.$$

In Homework 3 (7), we have showed that the center $Z(G)$ is a subgroup of G. Let H be a subgroup of G. Prove that the set

$$HZ(G) = \{hz \mid h \in H, z \in Z(G)\}$$

is a subgroup of G.

(6) (a) [**5 points**] What is the order of $([15]_{20}, [20]_{24})$ in $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$?

  (b) [**6 points**] What is the largest order of an element in $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$? And use your answer to show that $\mathbf{Z}_{20} \times \mathbf{Z}_{24}$ is not cyclic.

  (c) [**6 points**] Let $G = \mathbf{Z}_{10}^{\times} \times \mathbf{Z}_{10}^{\times}$. Let $H = \langle (3, 7) \rangle$ and $K = \langle (7, 7) \rangle$. Find HK in G. *Here,* $(3, 7)$ *means* $([3]_{10}, [7]_{10})$. *Just use this simplified notations in your answer.*