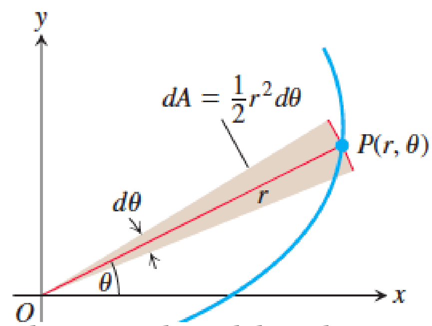
§ 11.5: Area/Length in Polar Coordinates

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \leq \theta \leq \beta$

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta}$$

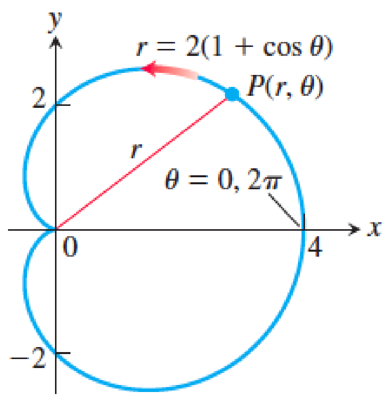


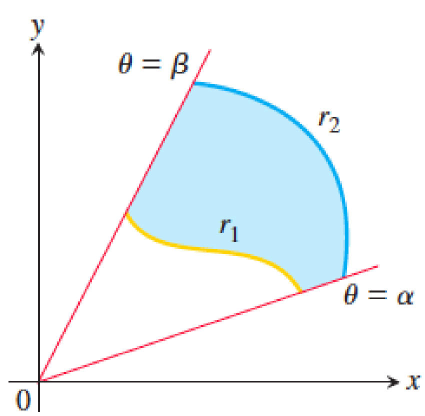
This is the integral of the **area differential**

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left(f(\theta) \right)^2 \, d\theta.$$

Example 1: Find the area of the region in the xy-plane enclosed by the cardioid

$$r = 2(1 + \cos\theta)$$

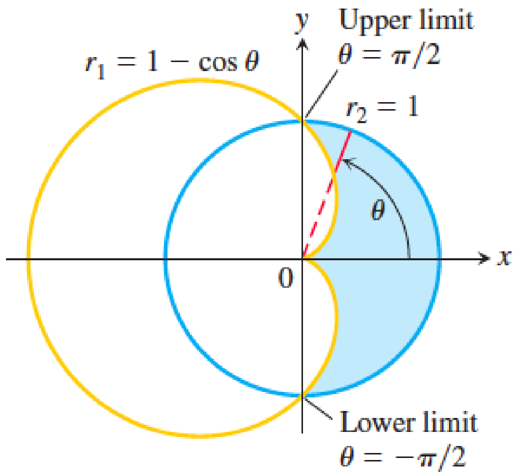




Area of the Region $0 \leq r_1(\theta) \leq r \leq r_2(\theta), \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \boxed{\int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta}$$

Example 2: Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.



If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the pt $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.}$$

Proof: $x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta, \quad \alpha \le \theta \le \beta$. Then we have

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 3: Find the length of the cardioid $r = 1 - \cos\theta$.