§ 11.2: Calculus with Parametric Curves

A parametrised curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if $f(t)$ and $g(t)$ are differentiable at t. At a point on a differentiable parametrised curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt and dy/dx are related by the

Chain Rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}.$

Parametric Formula for $\dfrac{dy}{dx}$: If all three derivatives exist and $\dfrac{dx}{dt} \neq 0$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Parametric Formula for $\dfrac{d^2y}{dx^2}$: Further we have $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(y') = \dfrac{dy'/dt}{dx/dt}$

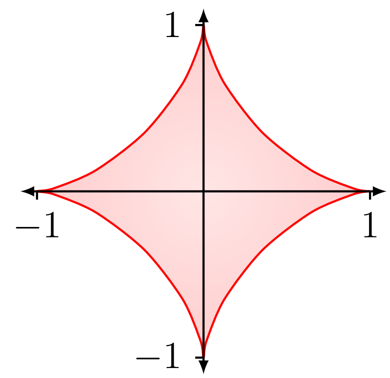
if y is a twice-differentiable function of x.

Example 1: Find the tangent to the curve $x = \sec(t)$, $y = \tan(t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, at the point $(\sqrt{2}, 1)$, where $t = \frac{\pi}{4}$.

Example 2: Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

Example 3: Find the area enclosed by the astroid

$$x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \le t \le 2\pi.$$

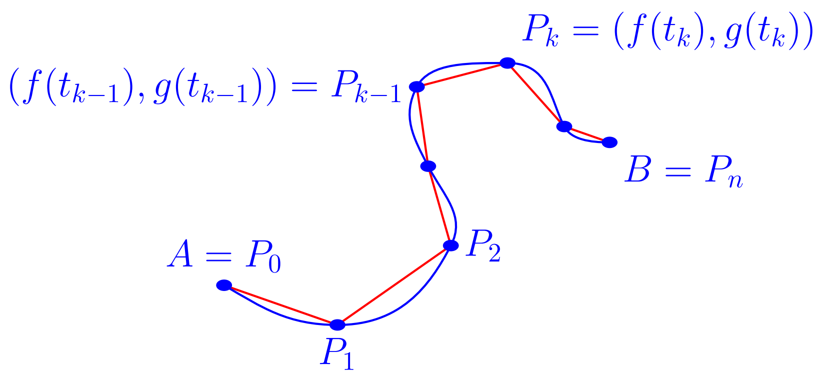


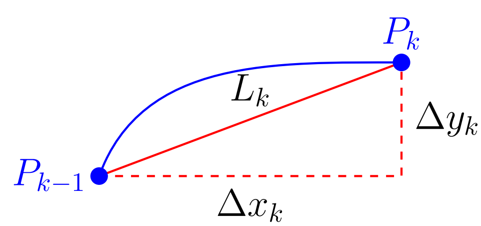
Let C be a curve given parametrically by the equations

$$x = f(t), \quad y = g(t), \quad a \le t \le b.$$

Assume that $f(t)$ and $g(t)$ are **continuously differentiable** on the interval $[a, b]$. And assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero, which prevents the curve C from having any corners or cusps. Such a curve is called a **smooth curve**. The **length of** C is the definite integral

$$\boxed{L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.}$$





Proof: The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1 and so on, ending at $B = P_n$.

The arc $P_{k-1}P_k$ is approximated by the straight line segment, which has length

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

We know by the Mean Value Theorem there exist numbers t_k^* and t_k^{**} that satisfy

$$f'(t_k^*) = \frac{f(t_k) - f(t_{k-1})}{\Delta t_k} \qquad \text{and} \qquad g'(t_k^{**}) = \frac{g(t_k) - g(t_{k-1})}{\Delta t_k},$$

thus the above becomes $\quad L_k = \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\, \Delta t_k. \quad$ Summing up each line segment we obtain an approximation for the length L of the curve C;

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\, \Delta t_k.$$

We obtain the desired exact value of L by taking a limit of this sum.

Example 4: Find the length of the circle of radius r defined parametrically by $x = r\cos(t), \quad y = r\sin(t), \quad 0 \leq t \leq 2\pi$.

Example 5: Find the length of the astroid $x = \cos^3(t), \quad y = \sin^3(t), \quad 0 \leq t \leq 2\pi$.

Definition: If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is traversed exactly once as t increases from a to b, then the **surface area of the surface of revolution** generated by revolving the curve about the coordinate axes are as follows.

1. **Revolution about the x-axis** $(y \geq 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. **Revolution about the y-axis** $(x \geq 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example 6: The standard parametrisation of the circle of radius 1 centred at the point $(0, 2)$ in the xy-plane is

$$x = \cos(t), \quad y = 2 + \sin(t), \quad 0 \leq t \leq 2\pi.$$

Use this parametrisation to find the surface area of the surface swept out by revolving the circle about the x-axis.