§ 10.10: Applications of Taylor Series

Evaluating Non-elementary Integrals: Taylor series can be used to express non-elementary integrals in terms of series.

Example 1: Express $\int \sin(x^2)\,dx$ as a power series.

Example 2: Estimate $\int_0^1 \sin(x^2)\,dx$ with an error of less than 0.001.

If we extend this to 5 terms, we obtain

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \frac{1}{19 \cdot 9!} \approx 0.310268303.$$

This gives an error of about 1.08×10^{-9}. To guarantee this accuracy (using the error formula) for the Trapezium Rule, we would need to use about 8000 subintervals!

Arctangents: In § 10.7, Ex 7, we found a series for $\tan^{-1} x$ by differentiating to get

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = 1 - x^2 + x^4 - x^6 + \cdots$$

and then integrating to get

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

But we didn't prove the term-by-term integration theorem on which this conclusion depended. We now derive the series again by integrating both sides of the finite formula

We take this route instead of finding the Taylor series directly because the formulas for the higher-order derivatives of $\tan^{-1} x$ are unmanageable.

Evaluating Indeterminate Forms: We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

Eg 3: Evaluate $\lim_{x\to 1} \frac{\ln x}{x-1}$. (*This limit can be evaluated using L'Hôpital's Rule as well.*)

Example 4: Evaluate $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

Euler's Identity: A complex number is a number of the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$. So

$$i=\sqrt{-1} \qquad i^2=-1 \qquad i^3=-\sqrt{-1} \qquad i^4=1 \qquad i^{4n+k}=i^k \qquad i^{2n+k}=(-1)^n i^k.$$

If we substitute $x=i\theta$ into the Taylor series for e^x and use the relations above, then

Euler's Identity: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, for any real number θ.

- You can use this identity to derive all of the angle sum formulas, so you never need to remember them all!
- This identity also enables us to define e^{a+bi} to be $e^a \cdot e^{bi}$ for any number $a + bi$.

Moreover, we see that $e^{i\pi} = -1$, which we can rewrite to obtain

$e^{i\pi} + 1 = 0$ which combines 5 of the most important constants in math; e, π, i, 1 and 0.

Common Taylor Series

1. $\dfrac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots$	$\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
2. $\dfrac{1}{1+x}$	$1 - x + x^2 - x^3 + \cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\lvert x\rvert < 1$
3. e^x	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$\lvert x\rvert < \infty$
4. $\sin(x)$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\lvert x\rvert < \infty$
5. $\cos(x)$	$1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} x^{2n}$	$\lvert x\rvert < \infty$
6. $\ln(1+x)$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} x^n$	$-1 < x \le 1$
7. $\tan^{-1}(x)$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$	$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} x^{2n+1}$	$\lvert x\rvert \le 1$