§ 10.8: Taylor and Maclaurin Series

Question: Can we also express functions of different forms as power series?

If we assume that a function $f(x)$ with derivatives of all orders is the sum of a power series about $x = a$ then we can readily solve for the coefficients c_n. Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

with positive radius of converges R. By repeated term-by-term differentiation within the interval of convergence, we obtain:

Definitions: Let $f(x)$ be a function with derivatives of all orders throughout some open interval containing a. Then the **Taylor Series generated by** $f(x)$ **at** $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin Series of** f is the Taylor series generated by f at $x = 0$, that is,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

Note: The Maclaurin series generated by f is often just called the Taylor series of f.

Example 1: Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 2$. Where does the series converge to $\frac{1}{x}$?

Definition: Let $f(x)$ be a function with derivatives of order $1, \ldots, N$ in some open interval containing a. Then for any integer n from 0 through N, the **Taylor polynomial of order** n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note: • Just as the linearization of $f(x)$ at $x = a$ provides the best linear approximation of $f(x)$ in a neighbourhood of a, the higher-order Taylor polynomials provide the "best" polynomial approximations of their respective degrees.

• We speak of a Taylor polynomial of *order* n rather than *degree* n because $f^{(n)}(a)$ may be zero. (See example below.)

Example 2: Find the Taylor Series and Taylor polynomials generated by $f(x) = \cos(x)$ at $x = 0$.

Example 3: Find the Maclaurin Series (Taylor series at $x = 0$) generated by $f(x) = \sin(x)$.

Example 4: Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.