§ 10.7: Power Series

Definition: A **power series about** $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A **power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Example 1: Taking all the coefficients to be 1 in the power series centered at $x = 0$ gives the **geometric power series** (first term $a = 1$ and ratio $r = x$):

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

One of the most important **questions** we can ask about a power series is

"for what values of x will the series converge?"

Since power series are functions, what we are really asking here is

"what is the **domain** of the power series?"

Example 2: Consider the power series

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = \sum_{n=0}^{\infty} \left(-\frac{x-2}{2}\right)^n$$

Ex 3: Use the **Ratio Test** to see for what values of x do the following series converge?

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

(b) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

The Convergence Theorem for Power Series: If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

$\rightsquigarrow$ A power series $\sum c_n (x-a)^n$ behaves in one of three possible ways:

It might converge $\begin{cases} \text{on some interval of } \textit{radius } R. \\ \text{everywhere.} \\ \text{only at } x = a. \end{cases}$

The Radius of Convergence of a Power Series: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a|>R$ but converges absolutely for x with $|x-a|<R$.
   As for the endpoints $x=a-R$ and $x=a+R$, we need to discuss them separately.
2. The series converges absolutely for every x (i.e., $R=\infty$).
3. The series converges only at $x=a$ and diverges elsewhere (i.e., $R=0$).

R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x=a$ is called the **interval of convergence**. The interval of convergence may be open, closed or half open, depending on the series at *endpoints*.

How to test a Power Series for Convergence:

1. Use Ratio (or Root) Test to find the interval where the series converges absolutely. Ordinarily, this is an open interval $|x-a|<R$ or $a-R<x<a+R$.
2. If $R<\infty$, test for convergence or divergence at each **endpoint** ($|x-a|=R$). (Comparison Test, Integral Test, Alternating Series Test.)

Example 4: Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}3^n}$.

Operations on Power Series: On the intersection of their intervals of convergence, two power series can be ***added*** and ***subtracted*** term by term just like series of constants. They can be ***multiplied*** just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important.

The Series Multiplication Theorem for Power Series: If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

We can also substitute a function $f(x)$ for x in a convergent power series:

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function $f(x)$ with $|f(x)| < R$.

For example: Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows that

$$\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$$

converges absolutely for $|4x^2| < 1$ or $|x| < \frac{1}{2}$.

The Term-by-Term Differentiation Theorem: If $\sum c_n (x-a)^n$ has radius of convergence $R > 0$, it defines a function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on $a - R < x < a + R$. Then this function $f(x)$ has derivatives of all orders inside the interval:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

$$\vdots$$

and so on. Each of these series converge at every point of the interval $a - R < x < a + R$.

Be Careful!! Term-by-Term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n^3 x)}{n^2} \qquad \textit{converges} \text{ for all } x.$$

But if we differentiate term by term we get the series

$$\sum_{n=1}^{\infty} n\cos(n^3 x) \qquad \text{which } \textit{diverges} \text{ for all } x.$$

This is **not** a power series since it is not a sum of positive integer powers of x.

Example 5: Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

Term-by-Term Integration Theorem: Suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a - R < x < a + R$. Then,

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \quad \text{converges for } a - R < x < a + R$$

$$\text{and} \quad \int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \quad \text{for } a - R < x < a + R.$$

Example 6: Given $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on $-1 < t < 1$, find a series representation for $f(x) = \ln(1+x)$.

Example 7: Identify the function $f(x)$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x < 1.$$