§ 10.6: The Alternating Series Test

Definition: A series whose terms alternate between positive and negative is called an **alternating series**. The n^{th} term of an alternating series is of the form

$$a_n = (-1)^{n+1} b_n \quad \text{or} \quad a_n = (-1)^n b_n, \qquad \text{where } b_n = |a_n| \text{ is a positive number.}$$

The Alternating Series Test: The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots, \qquad b_n > 0,$$

converges if the following two conditions are satisfied:

1. Nonincreasing: $b_n \geq b_{n+1}$ for all $n \geq N$, for some integer N,
2. $\lim_{n\to\infty} b_n = 0$.

Example 1: The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \qquad \text{(converges)}$$

Example 2: Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+16}$.

Recall that a convergent series that is not absolutely convergent is **conditionally convergent.**

If p is a positive constant, the sequence $\frac{1}{n^p}$ is a decreasing sequence with limit zero. Therefore, the ***alternating*** p***-series***

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad (p > 0) \qquad \text{(converges)}$$

Moreover, by the ordinary p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, which is convergent if $p > 1$ and divergent if $p \leq 1$, we conclude

- If $p > 1$, the alternating p-series converges absolutely.
- If $0 < p \leq 1$, the alternating p-series converges conditionally.

The Rearrangement Theorem for Absolutely Convergent Series: If $\sum a_n$ converges absolutely and $b_1, b_2, \ldots, b_n \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and $\sum b_n = \sum a_n$.

This (rearrangement) theorem doesn't hold for a ***conditionally convergent series.***

Convergence Tests for Series:

(1) Partial Sums (particularly with telescoping series) (§ 10.2)

(2) The nth-Term Test for Divergence (§ 10.2)

(3) **Geometric Series** Test & Geometric series sum formula (§ 10.2)

(4) p**-Series** Test (§ 10.3)

(5) Integral Test & Remainder Theorem for the Integral Test (§ 10.3)

(6) Direct Comparison Test & Limit Comparison Test (§ 10.4)

(7) Ratio Test & Root Test (§ 10.5)

(8) Alternating Series Test (§ 10.6)