§ 10.5: Absolute Convergence & Ratio/Root Tests

Ex 1: Consider the series $\sum\limits_{n=1}^{\infty} 5\left(-\frac{1}{4}\right)^{n-1}$.

Ex 2: Now consider $\sum\limits_{n=1}^{\infty} \left(-\frac{5}{4}\right)^{n-1}$.

The Absolute Convergence Test: If $\sum\limits_{n=1}^{\infty} |a_n|$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

Definitions: A series $\sum a_n$ **converges absolutely** (or is *absolutely convergent*) if the corresponding series of absolute values $\sum |a_n|$, converges.

Thus, if a series is absolutely convergent, it must also be convergent.

We call a series **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges. (e.g., $\sum\limits_{n=1} \frac{(-1)^n}{n}$; see next § 10.6.)

Example 3: Consider $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$.

The Ratio Test: Let $\sum a_n$ be any series and suppose

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** (*inconclusive*) about the series using this test.

Example 4: Use the Ratio Test to investigate the convergence of the following series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

The Root Test: Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

Then we have the following:

- If $L < 1$, then $\sum a_n$ converges absolutely.
- If $L > 1$ (including $L = \infty$), then $\sum a_n$ diverges.
- If $L = 1$, we can make **no conclusion** (*inconclusive*) about the series using this test.

Example 5: Use the Root Test to investigate the convergence of the following series.

1. $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

2. $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

3. $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Ex 6 (if time permits) Use the Ratio/Root Test to investigate the following series.

I) $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$ Ratio Test: $|a_{n+1}/a_n| \to \frac{1}{3} < 1$ converges absolutely

II) $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ Ratio Test: $|a_{n+1}/a_n| \to \infty$ diverges

III) $\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$ Root Test: $\sqrt[n]{|a_n|} \to \frac{4}{3} > 1$ diverges

IV) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{n+1}}$ Root Test: $\sqrt[n]{|a_n|} \to 0 < 1$ converges absolutely