§ 10.4: Comparison Tests for Series

In §8.8, we have already seen Direct/Limit Comparison Test for an improper integral.

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases} \quad \rightsquigarrow \quad p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$$

In this section, we will talk about Direct/Limit Comparison Test for series.

Problem 1: Determine if $\sum_{n=1}^{\infty} a_n$ converges, diverges, or no information.

If no information, then give (i) an example of a series that converges and (ii) an example of a series that diverges, both of which satisfy the given condition.

(a) If $0 \le a_n \le \dfrac{1}{n}$ for all n, we can conclude .

(b) If $\dfrac{1}{n} \le a_n$ for all n, we can conclude .

(c) If $0 \le a_n \le \dfrac{1}{n^2}$ for all n, we can conclude .

(d) If $\dfrac{1}{n^2} \le a_n$ for all n, we can conclude .

(e) If $\dfrac{1}{n^2} \le a_n \le \dfrac{1}{n}$ for all n, we can conclude .

Direct Comparison Test for Series: If $0 \le a_n \le b_n$ for all $n \ge N$, where $N \in \mathbb{N}$.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$; 2. If $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.

p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$; *geometric series* $\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \text{converges} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \ge 1 \end{cases}$

Example 3: Let $a_n = \frac{1}{2^n + n}$ and let $b_n = \left(\frac{1}{2}\right)^n$.

Example 4: Let $a_n = \frac{1}{n^2 + n + 1}$.

Ex 5: Use Direct Comparison Test to show that $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3 + n}}$ converges.

Unfortunately, the Direct Comparison Test doesn't always work like we wish it would. For example, let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2 - 1}$ for $n \ge 2$.

$\frac{1}{n^2} \le \frac{1}{n^2 - 1}$ $\Rightarrow$ Direct Comparison is inconclusive. Need The Limit Comparison Test!!

The Limit Comparison Test (LCT): Suppose that $a_n > 0$ and $b_n > 0$ for all $n \ge N$.

1. If $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Ex 6: Determine whether the series $\sum_{n=2}^{\infty} \frac{n^3 - 2n}{n^4 + 3}$ converges or diverges.

Ex 7: Determine whether the series $\sum_{n=1}^{\infty} \frac{10n + 1}{n(n+1)(n+2)}$ converges or diverges.

Ex 8: Determine whether the series $\sum_{n=1}^{\infty} \frac{1 + n \ln n}{n^2 + 5}$ converges or diverges.

Ex 9: Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges or diverges.