§ 10.3: The Integral Test

Non-decreasing Partial Sums: Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then the partial sums form a non-decreasing sequence since $S_{n+1} = S_n + a_{n+1}$.

Corollary Of Monotonic Sequence Theorem: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Consider the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$. (In § 10.2, we mention that this series diverges.)

Example 2: Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?

The Integral Test: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms. Suppose that there is a positive integer N such that for all $n \geq N$, $a_n = f(n)$, where $f(x)$ is a positive, continuous, decreasing function of x.

Then the series $\displaystyle\sum_{n=N}^{\infty} a_n$ and the integral $\displaystyle\int_N^{\infty} f(x)\,dx$ both converge or diverge.

Example 3: Show that the p-**series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, \quad \text{where } p \text{ is a real constant,}$$

converges if $p > 1$ and diverges if $p \leq 1$ ($p = 1$: *Harmonic Series*).

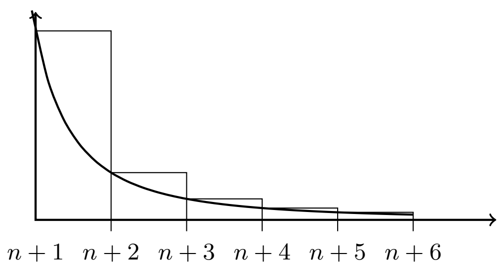
Example 4: Determine the convergence of divergence of the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

Error Estimation: For some convergent series, such as a geometric series or the telescoping series, we can actually find the total sum of the series. For most convergent series, however, we cannot easily find the total sum; see above **Examples 3 and 4**. Nevertheless, we can *estimate* the sum by adding the first n terms to get S_n, but we need to know how far off S_n is from the total sum S.

Suppose a series $\sum a_n$ is shown to be convergent by the integral test and we want to estimate the size of the remainder R_n measuring the difference between the total sum S and its n^{th} partial sum S_n.

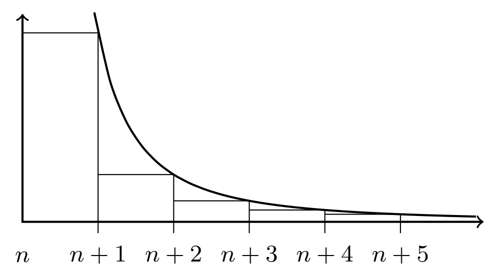
$$R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Lower Bound for R_n:



$$R_n \geq \int_{n+1}^{\infty} f(x)\,dx$$

Upper Bound for R_n:



$$R_n \leq \int_{n}^{\infty} f(x)\,dx$$

Bound for the Remainder in the Integral Test: $\boxed{\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx}$

Example 5: Estimate the sum S of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with $n = 10$.