§ 10.2: Infinite Series

Definitions: Given a sequence of numbers $\{a_n\}_{n=1}^{\infty}$, an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. a_n is the n^{th} **term** of the series. The sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

is called the **sequence of partial sums** of the series, S_n being the n^{th} **partial sum**.

If the sequence of partial sums converges to a limit L, we say that the series **converges** and that the **sum** is L. In this case we write

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums of the series does not converge, then the series **diverges**.

A **geometric series** is of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad \left(\qquad \qquad \qquad \right)$$

where a and r are fixed real numbers and $a \neq 0$. The **ratio** r can be positive or negative.

i) If $r = 1$, the n^{th} partial sum of the geometric series is

ii) If $r = -1$, the series diverges since the n^{th} partial sums alternate between a and 0.

iii) If $|r| \neq 1$, then we use the following "trick":

Example 2: Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$.

Example 3: Express the repeating decimal $5.232323\ldots$ as the ratio of two integers.

Example 4$^{!!}$: Find the sum of the **telescoping series**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: Suppose S_n converges to L. Then S_{n-1} also converges to L. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0.$$

The converse of this theorem is **not** true! (eg. in § 10.3: Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.)

The n^{th} Term Test for Divergence:

The series $\sum_{n=1}^{\infty} a_n$ *diverges* if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

i) $\sum_{n=1}^{\infty} n^2$ diverges since

ii) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges since

iii) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges since

Combining Series: If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent series, then

1) Sum/Difference Rule: $\sum_{n=1}^{\infty} (a_n \pm b_n)$

2) Constant Multiple Rule: $\sum_{n=1}^{\infty} ca_n$, for any $c \in \mathbb{R}$.

Caution! $\sum_{n=1}^{\infty} (a_n + b_n)$ can converge when both $\sum a_n$ and $\sum b_n$ diverge! $(a_n = n = -b_n)$

Example 5: Find the sums of the following series.

1. $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$

2. $\sum_{n=0}^{\infty} \frac{4}{2^n}$

Adding/deleting a **finite** number of terms will not alter the convergence or divergence.

eg., If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and conversely also true.