§ 10.1: Sequences

A **sequence** is a list of numbers written in a specific order. We *index* them with positive integers, $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$. These are the **terms** of the sequence.

A sequence may be *finite* or *infinite*. We will be looking specifically at *infinite* sequences which we will denote by $\{a_n\}_{n=1}^{\infty}$. Note that a_n is called the n**th term**.

Examples:

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$

(c) Fibonacci Sequence: (a *recursively defined sequence*)

$$\begin{cases} f_1 = 1 \\ f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3 \end{cases}$$

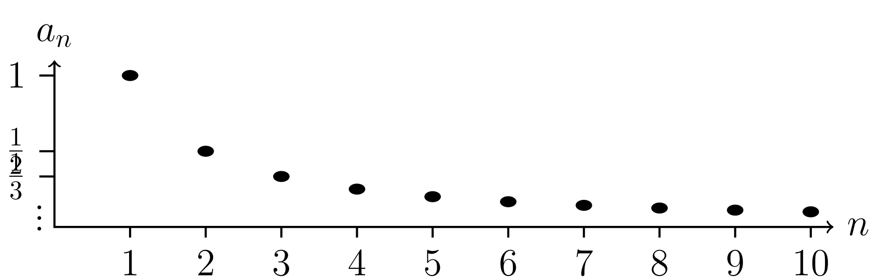
(**Precise Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if for every $\varepsilon > 0$ there exists an integer N such that

$$|a_n - L| < \varepsilon \qquad \text{for all } n \geq N.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

(**Friendly Definition of a Limit of a Sequence**) The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to the number L if $\lim_{n\to\infty} a_n = L$. If no such number L exists, we say that $\{a_n\}$ **diverges**.

Visualising a Sequence: Plot the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ in $\mathbb{R}^2$. What do you notice?



From the plot above it looks as if the sequence is tending towards 0.

Definition: $\lim_{n\to\infty} a_n = \infty$ means that for every positive integer M, there exists an integer N such that if $n \geq N$, then $a_n > M$.

Limit Rules for Sequences:

If $a_n \to L$, $b_n \to M$, then:

1. Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = \qquad ,$
2. Constant Rule: $\lim_{n\to\infty} c = \qquad$ for any $c \in \mathbb{R}$,
3. Product Rule: $\lim_{n\to\infty} a_n \cdot b_n = \qquad ,$
4. Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \qquad ,$ if $M \neq 0$
5. Power Rule: $\lim_{n\to\infty} a_n^p = \qquad ,$ if $p > 0$, $a_n > 0$

The Sandwich Theorem for Sequences: Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be three sequences such that there exists a positive integer N where

$$a_n \leq b_n \leq c_n, \quad \text{for each } n \geq N, \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim_{n\to\infty} b_n = L$.

Continuous Function Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ whenever n is a positive integer, then $\lim_{n\to\infty} a_n = L$.

Example: $f(x) = \frac{1}{x}$ satisfies $f(n) = a_n$ for every positive integer n, so $\lim_{n\to\infty} \frac{1}{n} = \lim_{x\to\infty} \frac{1}{x} = 0$.

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Examples of Convergent Sequences:

1. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

2. $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

3. $\left\{\frac{\cos(n)}{n}\right\}_{n=1}^{\infty}$

4. $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

Examples of Divergent Sequences: $\{(-1)^n\}_{n=1}^{\infty}$, $\{(-1)^n n\}_{n=1}^{\infty}$, $\{\sin(n)\}_{n=1}^{\infty}$.

Definition: n **factorial** $n! = n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. **Convention:** $0! = 1$.

Example 1: Find the limit of the sequence $\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$.

Example 2: For what values of r is the sequence $\{r^n\}_{n=1}^{\infty}$ convergent?

Commonly Occurring Limits:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

3. $\lim_{n\to\infty} x^{1/n} = 1$. ($x > 0$, fixed)

4. $\lim_{n\to\infty} x^n = 0$. ($|x| < 1$, fixed)

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. (any x, fixed)

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. (any x, fixed)

Definitions: Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequences and a *monotonic* sequence.

(a) A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}_{n=1}^{\infty}$.

If M is an upper bound for $\{a_n\}_{n=1}^{\infty}$ but no number less than M is an upper bound for $\{a_n\}_{n=1}^{\infty}$, then M is the **least upper bound (supremum)** of $\{a_n\}_{n=1}^{\infty}$.

(b) A sequence $\{a_n\}_{n=1}^{\infty}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}_{n=1}^{\infty}$.

If m is a lower bound for $\{a_n\}_{n=1}^{\infty}$ but no number greater than m is a lower bound for $\{a_n\}_{n=1}^{\infty}$, then m is the **greatest lower bound (infimum)** of $\{a_n\}_{n=1}^{\infty}$.

(c) If $\{a_n\}_{n=1}^{\infty}$ is bounded from above and below then $\{a_n\}_{n=1}^{\infty}$ is **bounded**.

If $\{a_n\}_{n=1}^{\infty}$ is not bounded, then we say that $\{a_n\}_{n=1}^{\infty}$ is an **unbounded** sequence.

(d) Every **convergent** sequence is **bounded**.

But **not** every bounded sequence converges. (consider ____________________).

(e) A sequence $\{a_n\}_{n=1}^{\infty}$ is **non-decreasing** if $a_n \leq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **non-increasing** if $a_n \geq a_{n+1}$ for every n.

A sequence $\{a_n\}_{n=1}^{\infty}$ is **monotonic** if it is either non-decreasing or non-increasing.

The Monotonic Sequence Theorem: Every bounded, monotonic sequence converges.

Note: The Monotonic Sequence Theorem **ONLY** tells us that the limit exists, **NOT** the value of the limit.

Example 3: Does the following recursive sequence converge?

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6).$$