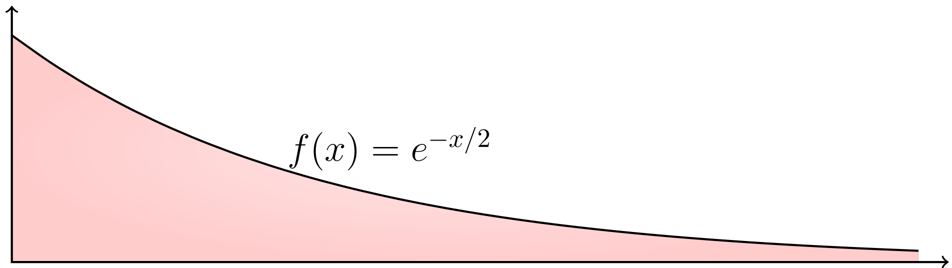
§ 8.8: Improper Integrals

Infinite Limits of Integration: Let's consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant.



Integrals with infinite limits of integration are called **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$.

2. If $f(x)$ is continuous on $(-\infty, b]$, then $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{a\to\infty} \int_{-a}^b f(x)\,dx$.

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$, where c is any real number.

In each case, if the limit is **finite** we say that the improper integral **converges** and the limit is the **value** of the improper integral. Otherwise, the improper integral **diverges**.

Example 1: Evaluate $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\,dx$.

Example 2: Evaluate $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

Example 3: For what values of p does the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge? When the integral does converge, what is its value?

Integrands with Vertical Asymptotes: Another type of improper integral that can arise is when the integrand has a vertical asymptote (infinite discontinuity) at a limit of integration or at a point on the interval of integration.

Example 4: Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Integrals of functions that become infinite at a point within the interval of integration are called **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$.

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$.

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say that the improper integral **converges** and the limit is the **value** of the improper integral. Otherwise, the improper integral **diverges**.

Example 5: Investigate the convergence of $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$.

Example 6: Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

Tests for Convergence: When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. The principal tests for convergence or divergence are the **Direct Comparison Test** and the **Limit Comparison Test**.

Direct Comparison Test for Integrals: If $0 \leq f(x) \leq g(x)$ on the interval $[a, \infty)$, where $a \in \mathbb{R}$, then,

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then so does $\displaystyle\int_a^\infty f(x)\,dx$.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then so does $\displaystyle\int_a^\infty g(x)\,dx$.

Example 6: Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2(x)}{x^2}\,dx.$$

Example 7: Determine if the following integral is convergent or divergent.

$$\int_3^\infty \frac{1}{x - e^{-x}}\,dx.$$

Limit Comparison Test for Integrals: If the positive functions $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or diverge.

Example 8: Show that $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,dx$ converges.

Example 9: Show that $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges.