§8.5: Integration by Partial Fractions

Method of Partial Fractions when $\dfrac{f(x)}{g(x)}$ **is proper (**$\deg f(x) < \deg g(x)$**)**

0. If the fraction is not proper, divide $f(x)$ by $g(x)$ and work with the remainder term.

1. Let $x-r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\boxed{\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \cdots + \frac{A_m}{(x-r)^m}}$$

   Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be an **irreducible** quadratic factor of $g(x)$ so that x^2+px+q has no real roots ($p^2-4q<0$). Suppose $(x^2+px+q)^n$ is the highest power of this factor dividing $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\boxed{\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \frac{B_3x+C_3}{(x^2+px+q)^3} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}}$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $\dfrac{f(x)}{g(x)}$ equal to the sum of all these partial fractions.
4. Solved for the undetermined coefficients by comparing coefficients of powers of x (or *strategically plugging in values*).

Example 1 (Distinct linear factors): Compute $\displaystyle\int \frac{x-7}{x^2-2x-3}\,dx$.

Exercise 1 (Practice for Ss): Find $\displaystyle\int \frac{x+15}{(3x-4)(x+1)}\,dx$.

$$\boxed{\frac{7}{3}\ln|3x-4| - 2\ln|x+1| + C}$$

Example 2 (Higher power of linear factor): Find $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx$.

Example 3 (Not Proper): Evaluate $\displaystyle\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$

Example 4 (Quadratic factor and linear factor): Compute $\displaystyle\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx.$

Exercise 2 (**Practice for Ss**): Evaluate $\int \frac{dx}{x(x^2+1)^2}$.

$$\boxed{\ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + C}$$

Exercise 3 (**Practice for Ss**): Evaluate $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$

$$\boxed{-\frac{1}{3}\ln|\cos\theta - 1| + \frac{1}{3}\ln|\cos\theta + 2| + C}$$