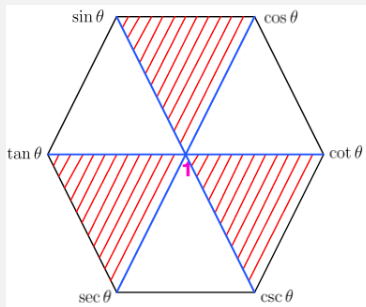
§ 8.3: Trigonometric Integrals

3 Diagonals: The product of two vertexes of the diagonal is 1.

3 Shadow Triangles: Sum of squares of top two vertexes equals to the square of bottom vertex.

Diagonals:

$$\sin\theta \cdot \csc\theta = 1$$
$$\tan\theta \cdot \cot\theta = 1$$
$$\sec\theta \cdot \cos\theta = 1$$



Shadow Triangles:

$$\sin^2\theta + \cos^2\theta = 1^2$$
$$\tan^2\theta + 1^2 = \sec^2\theta$$
$$1^2 + \cot^2\theta = \csc^2\theta$$

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Specially, let $x = y$ and we obtain the **double/half-angle formulas:**

$$\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \qquad \Rightarrow \cos^2(x) = \frac{1+\cos(2x)}{2} \\
&= 1 - 2\sin^2(x) \qquad \Rightarrow \sin^2(x) = \frac{1-\cos(2x)}{2}
\end{aligned}$$

Products of Powers of Sines and Cosines

$$\int \sin^m(x)\cos^n(x)\,dx, \quad \text{where } m \text{ and } n \text{ are non-negative integers.}$$

- If both m and n are even, we substitute (**Half-angle formulas**)

  $$\sin^2(x) = \frac{1-\cos(2x)}{2}, \qquad \cos^2(x) = \frac{1+\cos(2x)}{2},$$

  to reduce the integrand to one in lower power of $\cos(2x)$.

- Otherwise, we assume $m = 2k+1$ is odd (*similarly for n is odd*) and we write

  $$\sin^m(x) = \sin^{2k+1}(x) = \sin^{2k}(x)\cdot\sin(x) = (1-\cos^2(x))^k \cdot \sin(x).$$

  Then we make a u-substitution: $u = \cos(x)$.

Example 1: Find $\int \sin^3(x)\,dx$.

Example 2: Find $\int \sin^5(x)\cos^2(x)\,dx$.

Example 3: Find

$$\int \sin^2(x)\cos^4(x)\,dx.$$

Eliminating Square Roots

Example 4: Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos(4x)}\, dx$.

Integral of Powers of $\tan x$ and $\sec x$

Example 5: Evaluate

$$\int \tan^4(x)\, dx.$$

Example 6: Evaluate

$$\int \tan^4(x) \sec^4(x)\, dx.$$

Example 7*: Evaluate

$$\int \sec^3(x)\,dx.$$

- $\tan^2(x) + 1 = \sec^2(x)$
- $(\tan(x))' = \sec^2(x)$, $\quad \int \sec^2(x)\,dx = \tan(x) + C$
- $(\sec(x))' = \sec(x)\tan(x)$, $\quad \int \sec(x)\tan(x)\,dx = \sec(x) + C$