Final Exam Review

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Elementary Integration Formulas

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0, a \neq 1)$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

- Simplify the integrand if possible
- $\mathcal{U}$-substitution
- Integration by Parts: $\displaystyle\int U\,dV = UV - \int V\,dU$
- Trigonometric Substitution $\rightsquigarrow$ Trigonometric Integrals
- Integration by Partial Fractions $\dfrac{P(x)}{Q(x)}$
- Improper Integrals $\rightsquigarrow$ Application: Integral test for infinite series

Infinite Sequences

(i) Basic Limit Rules for Sequences: $(+,\ -,\ \times,\ \div$, power rule)

(ii) The Sandwich Theorem for Sequences

(iii) The Continuous Function Theorem for Sequences (L'Hôpital's Rule)

(iv) $$r^n \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{diverges} & \text{if } |r| > 1 \text{ or } r = -1 \end{cases}$$

(v) Commonly Occurring Limits

(vi) The Monotonic Sequence Theorem

Infinite Series

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n, \quad \text{where } S_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

Telescoping Series: e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

The *n*th Term Test for Divergence

If $\displaystyle\lim_{n\to\infty} a_n = L \neq 0$ or fails to exist, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Convergence Tests for Series

1. Integral test & *Remainder Theorem for the Integral Test*

   $a_n = f(n) \leftrightsquigarrow f(x)$ positive, continuous, decreasing: $\sum_{n=N}^{\infty} a_n \leftrightsquigarrow \int_N^{\infty} f(x)\,dx$

p-Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.$$

2. Direct/Limit Comparison test ($a_n > 0, b_n > 0$)

   (i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge/diverge.

   (ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

   (iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

3. Ratio/Root test $\left(\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L,\ \lim_{n\to\infty} \sqrt[n]{|a_n|} = L\text{: } L < 1 \leftrightsquigarrow \text{ abs. con.}\right)$

4. Alternating Series test: $b_n > 0, \sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges if $b_n \searrow 0$.

Power Series $\sum_{n=0}^{\infty} c_n (x-a)^n$

How to test a Power Series for Convergence:

1. Use **Ratio/Root Test** to find the interval where the series converges absolutely $\rightsquigarrow$ an open interval: $|x-a| < R$ or $a - R < x < a + R$
2. $R < \infty$: test for convergence/divergence at each endpoint $|x-a| = R$ (Comparison Test, Integral Test, Alternating Series Test, etc)

Operations on Power Series:

- $+, \quad -, \quad \cdot$ (on the intersection of their intervals of convergence)
- $\sum a_n x^n$ conv. abs. $|x| < R \quad \rightsquigarrow \sum a_n \big(f(x)\big)^n$ conv. abs. $|f(x)| < R$
- Term-by-Term Differentiation
- Term-by-Term Integration

Taylor Series Generated by $f(x)$ at $x = a$: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

- $a = 0$ $\rightsquigarrow$ Maclaurin (Taylor) Series of f
- Taylor polynomial of order n generated by f at $x = a$
- Applications of Taylor Series

1. $\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots$	$\sum_{n=0}^{\infty} x^n$	$\lvert x\rvert < 1$
2. $\frac{1}{1+x}$	$1 - x + x^2 - x^3 + \cdots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$\lvert x\rvert < 1$
3. e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\lvert x\rvert < \infty$
4. $\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\lvert x\rvert < \infty$
5. $\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$\lvert x\rvert < \infty$
6. $\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$	$-1 < x \le 1$
7. $\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$\lvert x\rvert \le 1$

Parameterized Plane Curves & Polar Coordinates

- Cartesian Equations **vs.** Parametric Equations & Converting
- Calculus with Parametric Curves
  1. Parametric Formula for First/Second Derivatives
  2. Arc Length of Smooth Curves: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
  3. Revolution about the x-axis: $S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
  4. Revolution about the y-axis: $S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
- Polar Coordinates (r, θ) $\leftrightsquigarrow$ $x = r\cos(\theta)$, $y = r\sin(\theta)$
  1. Area in Polar Coordinates: $A = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta$, $A = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2)\, d\theta$
  2. Arc Lengths in Polar Coordinates: $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$