Math 142—Exam II

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⋆ **No calculators** are allowed during this exam.

⋆ You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- **Indicate your final answer with a** $\boxed{\text{box}}$.

0. (Free 10 points) for taking the test. Enjoy!

1. [10 pts] Determine if the following statements are true or false: please **circle** true or false.

(a) True or False: If $\sum_{n=1}^{\infty} a_n$ is a series with $\lim_{n\to\infty} a_n = 0$, then the series converges.

$\sum_{n=1}^{\infty} \frac{1}{n}$. diverges.

(b) True or False: If a series converges, then that series converges absolutely.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$: Conditionally Converges.

(c) True or False: If $\sum_{n=1}^{\infty} a_n$ converges and $0 \le a_n \le b_n$ for all n, then $\sum_{n=1}^{\infty} b_n$ converges.

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1$: diverges.

(d) $\boxed{\text{True}}$ or False: If a series has a sequence of partial sums $\{S_n\}$ and $\lim_{n\to\infty} S_n = 1$, then the series converges to 1.

definition.

(e) $\boxed{\text{True}}$ or False: The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ converges.

Alternating Series Test

2. **[15 pts]** Which series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

(a) The geometric series $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

$r = -\frac{1}{4}$

$|r| < 1$ Converges.

$$S = \frac{a}{1-r} = \frac{5}{1-(-\frac{1}{4})} = \frac{5}{1+\frac{1}{4}} = \frac{5}{\frac{5}{4}} = \boxed{4}$$

(b) The telescoping series $\sum_{n=1}^{\infty} \underbrace{\Big(\ln(n) - \ln(n+1)\Big)}_{a_n}$

$$S(n) = \underbrace{\ln(1) - \cancel{\ln(2)}}_{a_1} + \underbrace{\cancel{\ln(2)} - \cancel{\ln(3)}}_{a_2} + \cancel{\cdots} + \underbrace{\cancel{\ln(n-1)} - \cancel{\ln(n)}}_{a_{n-1}} + \underbrace{\cancel{\ln(n)} - \ln(n+1)}_{a_n}$$

$$= -\ln(n+1) \longrightarrow -\infty \quad \text{as} \quad n \to \infty$$

$\Rightarrow$ $\boxed{\text{Diverges}}$

3. **[15 pts]** Determine if the following series **converge** or **diverge**. You must explicitly state the name of any test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+n+1}{n^4+n^2+1}$

Limit Comparison Test

$$\lim_{n\to\infty} \frac{\frac{n^3+n+1}{n^4+n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4+n^2+n}{n^4+n^2+1} = 1 > 0.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{n^3+n+1}{n^4+n^2+1}$ $\boxed{\text{diverges}}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Integral test: $f(x) > 0$, $\searrow$. Continuous.

$$\int_2^{\infty} \frac{1}{x \cdot \ln x}\, dx$$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$$= \lim_{b\to\infty} \int_{\ln 2}^{b} \frac{1}{u}\, du$$

$$= \lim_{b\to\infty} \ln|u| \Big]_{\ln 2}^{b}$$

$$= \lim_{b\to\infty} (\ln b - \ln \ln 2)$$

$= \infty$

$\Rightarrow \displaystyle\int_2^{\infty} \frac{1}{x \cdot \ln x}\, dx$ diverges

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\boxed{\text{diverges}}$

4. [**20 pts**] Determine if the following series **converge absolutely**, **converge conditionally**, or **diverge**. You must explicitly state the name of any test you use.

(a) $\sum_{n=1}^{\infty} \frac{n!}{(-3)^n(n+1)}$

$\boxed{\text{Ratio Test}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{(-3)^{n+1}\cdot(n+2)} \cdot \frac{(-3)^n\cdot(n+1)}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\cdot(n+1)}{3(n+2)} \right| = \infty > 1.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n!}{(-3)^n(n+1)} \boxed{\text{diverges!}}$$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

$\boxed{\text{Alternating Series Test}}$ $b_n = \frac{1}{\sqrt{n^2+1}} \searrow 0. \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ Converges.

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n^2+1}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

Using $\boxed{\text{Limit Comparison Test}}$ $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1 > 0$

the $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\Rightarrow \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n^2+1}}\right|$ diverges!

In Conclusion $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ $\boxed{\text{Conditionally Converges!}}$

5. **[15 pts]** Find the interval and radius of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+5}}.$$

(Hint: Apply the Ratio Test and the Additional Discussions for the Endpoints.)

I

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+5}}\cdot\frac{\sqrt{n^2+5}}{x^n}\right| = |x|\cdot\lim_{n\to\infty}\frac{\sqrt{n^2+5}}{\sqrt{n^2+2n+6}} = |x| < 1$$

$\Rightarrow \sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n^2+5}}$ absolutely converges on $|x|<1$, or $-1<x<1$

$\Rightarrow$ Radius of convergence is $\boxed{1}$

& interval of convergence is $\boxed{-1 \le x < 1}$

II

Endpoints: ① $x=-1$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n^2+5}}$

By Alternating Series Test. $b_n = \frac{1}{\sqrt{n^2+5}} \searrow 0$.

$\Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n^2+5}}$ Converges.

② $x=1$ $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+5}}$

Limit Comparison Test: $\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^2+5}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+5}} = 1 > 0.$

Again, Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. $\Rightarrow \sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+5}}$ diverges!

6. **[15 pts]** The series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

converges to $\sin x$ for all x. Using this to answer the following questions.

(a) Find a series for $\cos x$.

(b) For what values of x should the series in part (a) converge?

Note that $\cos x = (\sin x)'$

(a) $\Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cdot (2n+1)}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}} \cdots (*)$

By Term by Term Differential Theorem. $(*)$ holds $\boxed{\text{for all } x}$

(b) In fact: use Ration Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| = |x| \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0,$$

$\Rightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ $\boxed{\text{for all } x}$

Honor Statement: I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Printed Name: ______________________ Signature: ______________________

Problem	0	1	2	3	4	5	6	Total
Points	10	10	15	15	20	15	15	100
Score	10							