Exam II Review

Shaoyun Yi

MATH 142

University of South Carolina

Spring 2022

Infinite Series

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n, \quad \text{where } S_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

Telescoping Series: e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

The nth Term Test for Divergence

If $\displaystyle\lim_{n\to\infty} a_n = L \neq 0$ or fails to exist, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Convergence Tests for Series

1. Integral test & *Remainder Theorem for the Integral Test*

   $a_n = f(n) \rightsquigarrow f(x)$ positive, continuous, decreasing: $\displaystyle\sum_{n=N}^{\infty} a_n \rightsquigarrow \int_N^{\infty} f(x)\,dx$

p-Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \text{ and diverges if } p \le 1.$$

2. Direct/Limit Comparison test ($a_n > 0, b_n > 0$)
   - (i) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge/diverge.
   - (ii) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
   - (iii) If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
3. Ratio/Root test $\left(\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L, \lim_{n\to\infty} \sqrt[n]{|a_n|} = L\text{: } L < 1 \rightsquigarrow \text{ abs. con.}\right)$
4. Alternating Series test: $b_n > 0, \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges if $b_n \searrow 0$.

Power Series $\sum_{n=0}^{\infty} c_n (x-a)^n$

How to test a Power Series for Convergence:

1. Use **Ratio/Root Test** to find the interval where the series converges absolutely $\quad \rightsquigarrow$ an open interval: $|x-a| < R$ or $a - R < x < a + R$
2. $R < \infty$: test for convergence/divergence at each endpoint $|x-a| = R$ (Comparison Test, Integral Test, Alternating Series Test, etc)

Operations on Power Series:

- $+, \quad -, \quad \cdot \quad$ (on the intersection of their intervals of convergence)
- $\sum a_n x^n$ conv. abs. $|x| < R \quad \rightsquigarrow \sum a_n \big(f(x)\big)^n$ conv. abs. $|f(x)| < R$
- Term-by-Term Differentiation
- Term-by-Term Integration