Math 142—Exam I

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⋆ **No calculators** are allowed during this exam.

⋆ You are required to show your work on each problem on this exam. The following rules apply:

• **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

• **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

• **Indicate your final answer with a** $\boxed{\text{box}}$.

1. [15 points] Compute $\displaystyle\int \sin^4(x)\cos^5(x)\,dx$

$$= \int \sin^4(x)\cos^4(x)\cos(x)\,dx$$

$$= \int \sin^4(x)(1-\sin^2(x))^2 \cdot \cos(x)\,dx$$

$u = \sin(x)$. $du = \cos(x)\,dx$

$$= \int u^4(1-u^2)^2\,du$$

$$= \int u^4(1-2u^2+u^4)\,du$$

$$= \int u^8 - 2u^6 + u^4\,du$$

$$= \frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^9(x)}{9} - \frac{2\sin^7(x)}{7} + \frac{\sin^5(x)}{5} + C.}$$

2. [20 points] Find $\int e^x \cos(3x)\,dx$

$u = e^x.$ $\quad dv = \cos(3x)\,dx$

$du = e^x\,dx$ $\quad v = \frac{\sin(3x)}{3}$

$$= \frac{1}{3}e^x \cdot \sin(3x) - \int \frac{\sin(3x)}{3} \cdot e^x\,dx.$$

$$= \frac{e^x \cdot \sin(3x)}{3} - \frac{1}{3}\int e^x \cdot \sin(3x)\,dx.$$

$u = e^x$ $\quad dv = \sin(3x)\,dx$

$du = e^x\,dx$ $\quad v = -\frac{\cos(3x)}{3}$

$$= \frac{e^x \cdot \sin(3x)}{3} - \frac{1}{3}\left(-\frac{e^x \cdot \cos(3x)}{3} + \int \frac{\cos(3x)}{3} e^x\,dx\right)$$

$$\int e^x \cos(3x)\,dx = \frac{e^x \cdot \sin(3x)}{3} + \frac{e^x \cdot \cos(3x)}{9} - \frac{1}{9}\int e^x \cos(3x)\,dx$$

$$\Rightarrow \frac{10}{9}\int e^x \cos(3x)\,dx = \frac{e^x \cdot \sin(3x)}{3} + \frac{e^x \cos(3x)}{9} + C.$$

$$\Rightarrow \int e^x \cos(3x)\,dx = \boxed{\frac{1}{10}\left(3e^x \sin(3x) + e^x \cos(3x)\right) + C.}$$

3. **[20 points]** Find $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

$x = 4\cdot\sin\theta. \qquad dx = 4\cos\theta\, d\theta$

$$= \int \frac{16\cdot\sin^2\theta}{\sqrt{16-16\sin^2\theta}}\cdot 4\cos\theta\, d\theta$$

$$= \int \frac{16\cdot\sin^2\theta}{4\cos\theta}\cdot 4\cos\theta\, d\theta$$

$$= 16\cdot\int \frac{1-\cos(2\theta)}{2}\, d\theta$$

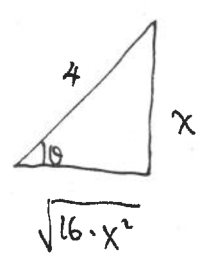
$$= 8\int \left(1-\cos(2\theta)\right) d\theta$$

$$= 8\left(\theta - \frac{\sin(2\theta)}{2}\right) + C$$

$$= 8\theta - 4\sin(2\theta) + C$$

$$= 8\theta - 8\sin\theta\cos\theta + C$$

$\frac{x}{4} = \sin\theta$



$$= 8\cdot\sin^{-1}\left(\frac{x}{4}\right) - 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C$$

$$= \boxed{8\cdot\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C}$$

4. [25 points] Solve $\displaystyle\int \frac{3x^3+7x^2+3x+5}{(x^2+1)(x+1)^2}\,dx$

$$\frac{3x^3+7x^2+3x+5}{(x^2+1)(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2}{(x^2+1)(x+1)^2}$$

$$\Rightarrow 3x^3+7x^2+3x+5 = A(x^3+x+x^2+1) + Bx^2+B + (Cx+D)(x^2+2x+1)$$

$$= Ax^3 + Ax^2 + Ax + A + Bx^2 + B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$

$$= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D)$$

$$\Rightarrow \begin{cases} 3 = A+C \Rightarrow C = 3-A \\ 7 = A+B+2C+D \\ 3 = A+C+2D \\ 5 = A+B+D \end{cases} \Rightarrow \begin{cases} 7 = A+B+2(3-A)+D \\ 3 = A+3-A+2D \\ 5 = A+B+D \end{cases} \Rightarrow \begin{cases} 1 = -A+B+D \\ 0 = 2D \Rightarrow D=0 \\ 5 = A+B+D \end{cases} \Rightarrow \begin{cases} 1 = -A+B \\ 5 = A+B \end{cases}$$

$$\Rightarrow B = A+1 \Rightarrow 5 = A+(A+1) = 2A+1 \Rightarrow 4 = 2A \Rightarrow A=2,\ B=3 \Rightarrow C=1.$$

$$\Rightarrow \int \frac{3x^3+7x^2+3x+5}{(x^2+1)(x+1)^2}\,dx = \int \frac{2}{x+1} + \frac{3}{(x+1)^2} + \frac{x}{x^2+1}\,dx$$

$$= \boxed{2\ln|x+1| - \frac{3}{x+1} + \frac{1}{2}\ln|x^2+1| + C}.$$

5. **[10 points]** Determine if the following improper interval converge or diverge. If the integral converges, determine what it converges to. Your answer should use proper notation.

$$\int_2^\infty xe^{-x^2}\,dx$$

$$= \lim_{a\to\infty} \int_2^a xe^{-x^2}\,dx$$

$u = -x^2$

$du = -2x\,dx$.

$$= \lim_{a\to\infty} \int_2^a e^{-x^2}\,d\left(-\tfrac{1}{2}x^2\right)$$

$$= \lim_{a\to\infty} -\tfrac{1}{2} \int_2^a e^{-x^2}\,dx^2.$$

$$= \lim_{a\to\infty} \left(-\tfrac{1}{2} e^{-x^2} \Big]_2^a \right)$$

$$= \lim_{a\to\infty} \left(-\tfrac{1}{2} e^{-a^2} + \tfrac{1}{2} e^{-4} \right)$$

$$= \boxed{\tfrac{1}{2} e^{-4}}.$$

6. **[10 points]** Which of the sequences $\{a_n\}_{n=1}^{\infty}$ converge, and which diverge? Find the limit of each convergent sequence.

(a) $a_n = \dfrac{4n^3 - 1}{n^3 + 8n}$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{4x^3-1}{x^3+8x} = \boxed{4.}$$

(b) $a_n = \dfrac{\sin n}{n^2}$

$$\underbrace{-\frac{1}{n^2}}_{n\to\infty\ \to\ 0} \le \frac{\sin n}{n^2} \le \underbrace{\frac{1}{n^2}}_{n\to\infty\ \to\ 0}$$

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{x\to\infty} \frac{1}{x^2} = 0.$$

$$\Rightarrow \quad \lim_{n\to\infty} a_n = \boxed{0}$$

Honor Statement: I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Printed Name: ____________________ Signature: ____________________

Problem	**1**	**2**	**3**	**4**	**5**	**6**	Total
Points	15	20	20	25	10	10	100
Score							