Exam I Review

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Elementary Integration Formulas

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0, a \neq 1)$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

Simplify the integrand if possible

Example

$$\int \sqrt{x}(1+\sqrt{x})\,dx = \int (\sqrt{x}+x)\,dx = \cdots$$

Example

$$\int \frac{\tan\theta}{\sec^2\theta}\,d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\,d\theta = \int \sin\theta\cos\theta\,d\theta = \cdots$$

$\mathcal{U}$-substitution

$\mathcal{U} = g(x)$ is in the integrand and its differential $d\mathcal{U} = g'(x)\,dx$ also occurs.

Example

$$\int x^2 e^{x^3}\,dx, \qquad \mathcal{U} = x^3, \quad d\mathcal{U} = 3x^2\,dx$$

Example

$$\int \frac{\ln x}{x}\,dx, \qquad \mathcal{U} = \ln x, \quad d\mathcal{U} = \frac{1}{x}\,dx$$

Integration by Parts: $\int U\,dV = UV - \int V\,dU$

Usually **two different types of functions** show up at the same time.

Example

$$\int x\sin x\,dx, \qquad U = x, \quad dV = \sin x\,dx$$

Example

$$\int x^2 e^x\,dx, \qquad U = x^2, \quad dV = e^x\,dx \quad (\textbf{Twice}\ \text{I.B.P.})$$

Example

$$\int x\ln x\,dx, \qquad U = \ln x, \quad dV = x\,dx$$

Example

$$\int e^x\sin x\,dx, \qquad U = \sin x, \quad dV = e^x\,dx \quad (\textbf{Twice}\ \text{I.B.P.})$$

Trigonometric Integrals

(i) Common Trig. Identities & Half/Double Angle Formulas

(ii) $\int \sin^m(x)\cos^n(x)\,dx$ & $\int \sec^m(x)\tan^n(x)\,dx$

Example

$$\int \sin^2(x)\cos^2(x)\,dx = \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,dx$$

Example

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx, \qquad u=\cos x, \quad du=-\sin x\,dx$$

Example

$$\int \tan^2(x)\sec^2(x)\,dx, \qquad u=\tan(x), \quad du=\sec^2(x)\,dx$$

Trigonometric Substitution

1. $\sqrt{a^2 - x^2}$, $\quad x = a\sin\theta$ and use Identity $1 - \sin^2\theta = \cos^2\theta$.

$$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx, \qquad x = 3\sin\theta, \quad dx = 3\cos\theta\,d\theta$$

2. $\sqrt{a^2 + x^2}$, $\quad x = a\tan\theta$ and use Identity $1 + \tan^2\theta = \sec^2\theta$.

$$\int \frac{1}{x^2\sqrt{x^2 + 4}}\,dx, \qquad x = 2\tan\theta, \quad dx = 2\sec^2\theta\,d\theta$$

3. $\sqrt{x^2 - a^2}$, $\quad x = a\sec\theta$ and use Identity $\sec^2\theta - 1 = \tan^2\theta$.

$$\int \frac{1}{\sqrt{x^2 - 4}}\,dx, \qquad x = 2\sec\theta, \quad dx = 2\sec\theta\tan\theta\,d\theta$$

Integration by Partial Fractions $\frac{P(x)}{Q(x)}$

1. If $\deg(P(x)) \geq \deg(Q(x))$, do the **long division** first.
2. Factor the denominator $Q(x)$ as far as possible.

Linear factors:

$$(x-r)^n \quad \leadsto \quad \sum_{i=1}^{n} \frac{A_i}{(x-r)^i}$$

Irreducible quadratic factors

$$(x^2+px+q)^m, \text{ where } p^2-4q<0 \quad \leadsto \quad \sum_{j=1}^{m} \frac{B_j x + C_j}{(x^2+px+q)^j}$$

Improper Integrals

Example

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \cdots$$

Example

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx = \cdots$$

Sometimes, L'Hôpital's Rule is helpful to evaluate the limits.

Test for Convergence: *Direct*/*Limit* Comparison Test

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$$

Infinite Sequences

(i) Basic Limit Rules for Sequences: $(+,\ -,\ \times,\ \div$, power rule$)$

(ii) The Sandwich Theorem for Sequences

(iii) The Continuous Function Theorem for Sequences (L'Hôpital's Rule)

(iv) $$r^n \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{diverges} & \text{if } |r| > 1 \text{ or } r = -1 \end{cases}$$

(v) Commonly Occurring Limits

(vi) The Monotonic Sequence Theorem

Infinite Series

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n, \quad \text{where } S_n := \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Geometric Series: $\displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$

Telescoping Series: e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Its inverse is not true in general. e.g. (§ 10.3) **Harmonic Series:** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

The *n*th Term Test for Divergence

If $\lim_{n\to\infty} a_n = L \neq 0$ or fails to exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.