Review for Final

Shaoyun Yi

MATH 122

University of South Carolina

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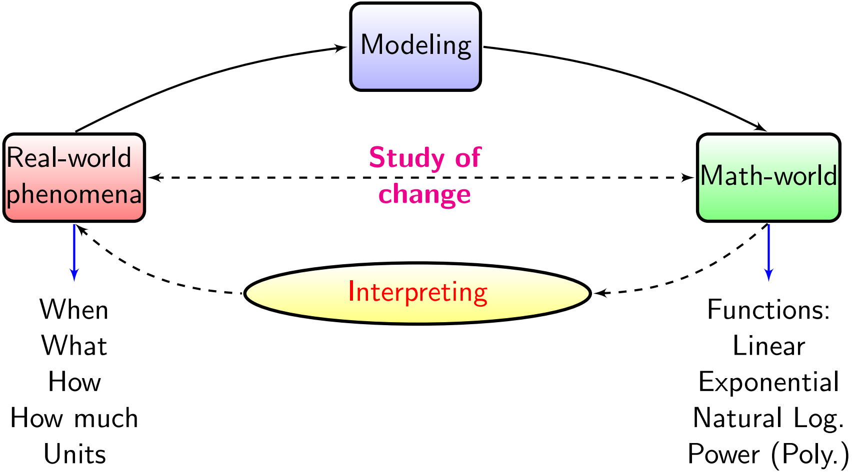

- Average R.O.C. $\xrightarrow{\Delta t \to 0}$ R.O.C of $f(t)$ at $t = a$ (*Derivative* $f'(a)$).
- Accumulated Change $\xrightarrow{\Delta t \to 0}$ Definite Integral $\int_a^b f(t)\,dt$ (**F.T.C.**).
- https://people.math.sc.edu/shaoyun/Review_122F_SYi_Fa_21.pdf
- Composite Functions $\leftrightsquigarrow$ Chain Rule & Integration by Substitution

Functions

Ask yourself:
- What is a function? $\rightsquigarrow$ domain (input)/range (output)
- Difference between closed/open intervals, i.e., $[,]$ / $(,)$
- Interpret your numerical answers correctly
- Horizontal intercepts (Zeros) & Vertical intercept $(0, f(0))$
- Function behavior: increasing/decreasing/constant
- Concavity: concave up/down/neither (a line)

- **Linear:** $y = mx + b$ & $y - y_0 = m(x - x_0)$
- **Exponential:** $P = P_0 \cdot a^t \overset{\text{base change}}{\longleftrightarrow} P = P_0 \cdot e^{kt}$
  - P_0 : initial quantity (vertical intercept)
  - r : decimal representation of percent rate of change ($r = a - 1$)
  - k : continuous rate of change
  - base change: $a = e^k \Leftrightarrow \ln a = k$
  - $a > 1$ $(r > 0, k > 0)$: exponential growth
  - $1 > a > 0$ $(r < 0, k < 0)$: exponential decay
- **Natural Log.:** $y = \ln x \Leftrightarrow e^y = x \quad (x > 0)$
- **Power (Poly.):** $Q(x) = k \cdot x^p$ $(P_n(x) = c_n x^n + \cdots + c_1 x + c_0, c_n \neq 0)$
- **Composite Functions:** $f(g(x))$ Realize the inside function $u = g(x)$

Changes

1. Change in y : $\Delta y = y_2 - y_1$ $\overset{\text{unit}}{\rightsquigarrow}$ unit of y
2. Average Rate of Change: $\overset{\text{unit}}{\rightsquigarrow}$ unit of y per unit of t

$$\frac{\Delta y}{\Delta t} = \frac{f(b) - f(a)}{b - a} = \text{Slope of secant line between } t = a \text{ and } t = b$$

- **Linear:** Its slope m is the constant Average Rate of Change.
- $\Delta t \to 0$: (Instantaneous) Rate of Change

$$\boxed{f'(a) = \text{Slope of tangent line at } t = a} \qquad (\star)$$

  - Estimate $f'(a)$ by taking $\Delta t = 0.001$ in Average Rate of Change
  - Determine $f'(a)$ is $\begin{matrix} > \\ < \\ = \end{matrix}$ 0 by reading the graph and using $(\star)$
  - Estimate $f'(a)$ given numerically (usually, right-hand approximation)
  - **First Derivative Test:** $f' > 0$ means $f \nearrow$ **v.s.** $f' < 0$ means $f \searrow$

3. Relative Change in P : $\dfrac{P_1 - P_0}{P_0}$ $\overset{\text{unit}}{\rightsquigarrow}$ %

Applications

- Average Rate of Change: Average Velocity $= \dfrac{\text{change in distance}}{\text{change in time}}$
- **Linear:** Cost $C(q)$, Revenue $R(q)$, Profit $\pi(q)$
  - $C(q) = C_{\text{fixed}} + C_{\text{variable}}$
  - $R(q) = p \cdot q$
  - $\pi(q) = R(q) - C(q)$
  - Break-even point, i.e., zero of $\pi(q)$
  - Marginal (R.O.C.) Cost/Revenue/Profit (Slopes of linear functions)
- **Exponential:** Know how to use "$a = 1 + r$" and "e^k" properly
  - Exponential growth: has a constant Doubling Time
  - Exponential decay: has a constant Half-Life
  - Compound interest: $\begin{cases} \textit{annually} & P(t) = P_0 \cdot (1 + r)^t \\ \textit{continuously} & P(t) = P_0 \cdot e^{rt} \end{cases}$
- **Natural Log.:** Solve exponential equations using Natural Logarithm
- **Power:** Define a polynomial (sum of power functions)
- **Composite Functions:** Produce new functions from old

How to Use Calculator

1. Graph a Function:

   (i) [$y =$] (ii) [window]: X_{min}/X_{max} (iii) [zoom]: Choose "ZoomFit"

2. Plot a Table of Data:

   (i) [$y =$]: plot1 (ii) [stat]: EDIT & 1: (iii) [zoom]: Choose "ZoomStat"

3. Find a Maximum:

   (i) [$y =$]: (ii) [window]: X_{min}/X_{max} (iii) [zoom]: Choose "ZoomFit"
   (iv) 2nd + [trace]: Choose "maximum" (v) Left/RightBound?/Guess?

4. Evaluate a Value $Y_1(X)$:

(i) [$y =$] (ii) 2nd + [mode] main screen (iii) [vars] Y-VARS 1: 1: Y_1 (iv) $Y_1(X)$

Formulas for Derivatives (*PDF is in Blackboard*)

1. **Constant Rule:** $(k)' = 0$
2. **Power Rule:** $(x^n)' = n\,x^{n-1}$
3. **Exponential Rule:** $(a^x)' = (\ln a)\,a^x$
4. **Exponential Rule:** $(e^x)' = e^x$
5. **Exponential Rule:** $(e^{kx})' = k\,e^{kx}$
6. **Natural Logarithmic Rule:** $(\ln x)' = \dfrac{1}{x}$
7. **Constant Multiple Rule:** $(c \cdot f)' = c \cdot f'$
8. **Sum Rule:** $(f + g)' = f' + g'$
9. **Difference Rule:** $(f - g)' = f' - g'$
10. **Product Rule:** $(f \cdot g)' = f' \cdot g + f \cdot g'$
11. **Quotient Rule:** $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$
12. **Chain Rule:** $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

(Instantaneous) Rate of Change/Derivative

1. Leibniz's Notation: $f'(x) = \dfrac{dy}{dx}$ and $f'(a) = \left.\dfrac{dy}{dx}\right|_{x=a}$
2. Interpretations: WWHHmU
3. Tangent Line Approximation: $f(x) \approx f(a) + f'(a) \cdot (x - a)$
4. Relative rate of change of y at a is $\dfrac{f'(a)}{f(a)}$ and Unit is % per unit of t
5. **1st derivative test**: $f' > 0$ means $f \nearrow$ **v.s.** $f' < 0$ means $f \searrow$
6. **2nd derivative test**: $f'' > 0$ means f 🙂 **v.s.** $f'' < 0$ means f 🙁
7. All derivative rules: *C, Power/Exp/NLog, CM, S/D, Prod./Quot., Chain*
   - Find Equation of tangent line: $m = f'(a)$ & tangent point $(a, f(a))$
   - Find Second derivative: $f'' = (f')'$
   - Find Marginal Revenue/Cost/Profit: e.g. $MR = R'(q)$, MC, MP.
   - Find Specific values: Review *Examples in §3.3 and §3.4*
8. Local Maxima/Minima**:** Critical points ($f'(p) = 0$) & f' sign changes
   Method: **2nd derivative test** ($f''(p) \neq 0$); otherwise, **1st derivative test**
9. Global Maxima/Minima**:** Compare critical values and endpoints values
   Application in real life: Find Maximum/Minimum profit ($MP = \pi'(q) = 0$)
10. Inflection points (f changes concavity)**:** $f''(p) = 0$ & f'' sign changes

Definite Integral

$$\begin{cases} \text{Left-hand sum} = f(t_0)\cdot\Delta t + f(t_1)\cdot\Delta t + \cdots f(t_{n-1})\cdot\Delta t \\ \text{Right-hand sum} = f(t_1)\cdot\Delta t + f(t_2)\cdot\Delta t + \cdots f(t_n)\cdot\Delta t \end{cases}$$

$$\Delta t = \frac{b-a}{n}$$

Definite integral $\int_a^b f(t)\,dt := \lim_{n\to\infty} \left(\text{Left/Right-hand sum}\right)$ (i.e., $\Delta t \to 0$)

- *Use a calculator:* math & choose "9." *when you know* f.
- Geometric side:
  - If $f(t) > 0$, then $\int_a^b f(t)\,dt =$ area under graph of f
  - If $f(t) < 0$, then $\int_a^b f(t)\,dt = -$ area between a and b
  - Area between two curves
- Estimate a definite integral *numerically/graphically*: **Left-/Right- sum**
- Interpretation: unit for $\int_a^b f(t)\,dt =$ product of unit of f and unit of t
- **The Fundamental Theorem of Calculus:** $\int_a^b F'(x)\,dx = F(b) - F(a)$

e.g. Total cost $C(b)$ of producing b units: $C(b) = C(0) + \int_0^b C'(q)\,dq$.

Indefinite Integral & Use FTC to evaluate definite integrals

1. Antiderivative: If $F'(x) = f(x)$, then $F(x)$ is an *antiderivative* of $f(x)$.

   Analyzing antiderivative graphically (FDT, FTC) & numerically (Calculator)

2. The Indefinite Integral of $f(x)$: $\displaystyle\int f(x)\,dx = F(x) + C$
   - The *family* of antiderivatives of $f(x)$;
   - **Formulas for Antiderivatives:** $f(x) = k$; $x^n\,(n \neq -1)$; $\dfrac{1}{x}$; e^x; e^{kx};

     Properties of Antiderivatives: Sums & Constant Multiples
   - Integration by substitution **(u-substitution):** $u = u(x)$, $du = u'(x)\,dx$

3. Using **FTC** to evaluate Definite Integrals:

   (i) Find an antiderivative $F(x)$, i.e., Calculate $\displaystyle\int f(x)\,dx = F(x) + C$

   (ii) Evaluate the definite integral: $\displaystyle\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

   (iii) A warm tip: *You can use a calculator to double-check your answers!*

   (iv) Applications: • Find the area; • Evaluate definite integrals by u-**sub**.

i) **Constant Rule:** $(k)' = 0$

ii) **Power Rule:** $(x^n)' = n\,x^{n-1}$

iii) **Exponential Rule:** $(a^x)' = (\ln a)\,a^x$

iv) **Exponential Rule:** $(e^x)' = e^x$

v) **Exponential Rule:** $(e^{kx})' = k\,e^{kx}$

vi) **Natural Logarithmic Rule:** $(\ln x)' = \frac{1}{x}$

vii) **Constant Multiple Rule:** $(c \cdot f)' = c \cdot f'$

viii) **Sum Rule:** $(f+g)' = f' + g'$

ix) **Difference Rule:** $(f-g)' = f' - g'$

x) **Product Rule:** $(f \cdot g)' = f' \cdot g + f \cdot g'$

xi) **Quotient Rule:** $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

xii) **Chain Rule:** $(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$

i) $\int k\,dx = kx + C$

ii) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

iii) $\int a^x\,dx = \frac{a^x}{\ln a} + C$

iv) $\int e^x\,dx = e^x + C$

v) $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$

vi) $\int \frac{1}{x}\,dx = \ln|x| + C$

vii) $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$

viii) $\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$

ix) $\int \big(f(x) - g(x)\big)\,dx = \int f(x)\,dx - \int g(x)\,dx$

x) **NA**

xi) **NA**

xii) **u-substitution:** $du = u'(x)\,dx$

Additional Suggestions

Review:

1. Your WileyPlus Homework
2. Your Class Notes
3. Quizzes with Solutions in Blackboard
4. Tests 1-3 Review Problems with Solutions in Blackboard
5. Tests 1-3 with Solutions in Blackboard

Contact:

- Me (office hours or by e-mail)

Good luck with all finals!